High Order Integral Nested Sliding Mode Control

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***Abstract*— This paper exposes a controller for the nonlinear systems in the block controllable form. This proposal guarantees exponential exact tracking in the presence of unknown matched and unmatched disturbances by means a combination of the block control method and integral terms designed with high-order sliding-modes algorithms. Both, matched and unmatched, disturbances are compensated by those integral continuous terms and the tracking is achieved with the design of a nominal control law.**

I. INTRODUCTION

A basic problem in the design of feedback control systems in the stabilization and tracking in presence of uncertainty caused by plant parameter variations and external perturbations. In order to deal with these problems, several approaches have been proposed. Most of them are based on Lyapunov stability theory and variable structure systems with sliding modes (SM). The SM techniques are based on the idea of the sliding manifold, that is an integral manifold with finite reaching time [1] and have been widely used for the problems of dynamic systems control and observation due to their characteristics of finite time convergence, robustness and insensitivity to uncertainties due to external bounded disturbances and parameters variation [2].

For the design process of controllers based on Lyapunov and SM it is recognized the disturbances which belongs to the control subspace, the matched ones, and those disturbances that appear into a subspace spanned by different than the control coordinates, the unmatched. For the sliding mode control case, the closed-loop system can be proposed to be insensitive to a certain class disturbances, which results to be the matched disturbances [3]. However, these controllers are not able to compensate the disturbances affecting the motion on the sliding manifold, i.e. unmatched disturbances, making the controller synthesis for the systems with unmatched disturbances a high challenging and interesting problem.

For systems presented in some block-wise form as the regular form [4], or block controllable forms [5], [6], the design procedures is performed, usually by applying step-by-step algorithms as the block control or the back-stepping [7], using some of the states as intermediate control variables or pseudo-controllers. Also, these forms allow the easy identification of the matched and the unmatched disturbances, leading to a straight calculation of pseudo-controllers which reduce the unmatched disturbances effect.

Taking into account that, for the case of SM control, the pseudo-control proposal is derived from the sliding manifold design, a manifold with high gain structure to attenuate the unmatched disturbances and to stabilize the SM dynamics for systems presented in the nonlinear block controllable (NBC) form [6] is proposed in [8]. Similarly, for this class of systems, the nested SM control [9] is proposed by replacing the high gain terms with sigmoid functions with the aim to create a quasi-sliding dynamics. Note that, in order to induce the SM dynamics, the manifold must be differentiable, this is the reason for the use of sigmoid functions instead of the well-known sign function.

As alternative to the mentioned high gain methods, a common and effective approach is the design of sliding manifolds which include integral SM control terms as, for example, the proposed in [10]–[12], including high order designs [13]. The integral SM control [14]–[16] has been proposed with the aim to force the system trajectory starting from the sliding manifold, eliminating the reaching phase and ensuring robustness. These controllers have been shown high performance and easy implementation, specially with its discrete version as shown in [17]–[20]. An important case of the application of integral SM control terms is the use of integral nested SM algorithms (IN-SM) [21], [22], which are based on the application of the nested SM control, combined with the integral SM control to systems presented in the NBC form. In this way, the motion on the sliding manifold has the characteristics of the integral SM controllers, rejecting or attenuating the unmatched disturbances. However, as for the case of nested SM control, avoiding the sliding manifold to contain discontinuous terms as the sign function, continuous approximation of the sign function is applied in the IN-SM case, where the sign function is replaced by a sigmoid function for each block. This proposal allows the design of a well defined manifold, but, with reduced robustness and tracking performance.

In order to overcome this major drawback of the IN-SM scheme, in this paper a new control algorithm for systems presented in the NBC form, the integral nested high order sliding mode control (IN-HOSM). The first forms of IN-HOSM were presented in [23], [24] with the main idea to use the quasi-continuous SM (QC-SM) algorithms [25] instead

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of sigmoid functions (used by the IN-SM) to the integral SM terms design, leading to a nested integral structure but with exact disturbance rejection. A similar technique presented in [26], which is an improvement of the mentioned high gains methods, offering exact tracking an finite time convergence. It is worth to highlight that the QC-SM algorithms can be designed to be differentiable for each block by selecting a suitable order.

The closed-loop system exhibits the properties of exponential tracking and robustness, rejecting the uncertainty due to the parameter variations and external disturbances.

The paper is exposed as follows: Section II presents the nonlinear system in the NBC form to be studied and the problem formulation. Section III describes the proposed controller, including a stability and robustness analysis. Simulation results which demonstrate the main characteristics of the proposed controller, are presented in Section IV. Finally, in Section V the conclusions are given.

II. PROBLEM STATEMENT

Consider the nonlinear system in the NBC form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,t) + B_1(x_1,t)x_2 + \Delta_1(x_1,t)\\ &\vdots\\ \dot{x}_i &= f_i(\bar{x}_i,t) + B_i(\bar{x}_i,t)x_{i+1} + \Delta_i(\bar{x}_i,t)\\ &\vdots\\ \dot{x}_r &= f_r(x,t) + B_n(x,t)u + \Delta_r(x,t)\end{aligned} \tag{1}$$

where $i = 2,\ldots,r-1$, $t \geq 0$ is the time variable, $x = \begin{bmatrix} x_1^T & \ldots & x_r^T \end{bmatrix}^T$ is the system state, divided in r blocks $x_i \in \mathcal{X}_i \subset \mathbb{R}^{n_i}$, with the hierarchical structure $\bar{x}_i = \begin{bmatrix} x_1^T & \ldots & x_i^T \end{bmatrix}^T$ and $n_1 \leq n_2 \leq \ldots \leq n_r$, $u \in \mathbb{R}^{n_r}$ is the control input, $f_i(\cdot)$ are known nonlinear smooth vector fields, $B_i(\bar{x}_i,t) \in \mathbb{R}^{n_i \times n_1}$ are known full rank and uniformly bounded matrices in $\mathcal{X}_i$ and, $\Delta_i(\bar{x}_i,t)$ are unknown bounded perturbation terms due to parameter variations and external disturbances.

The control output is $y = x_1$, the system state x and the control signal u are assumed to be known. The control problem is to design a controller such that the output tracks a smooth desired reference y_d in spite of the perturbations presence.

III. CONTROLLER DESIGN

A. On the Quasi-continuous controller

Consider the following SISO affine system:

$$\dot{\phi} = a(t,\phi) + b(t,\phi)u \tag{2}$$

where $\phi \in \mathbb{R}^p$. The control objective is the finite time stabilization of a variable $\sigma(t,\phi) \in \mathbb{R}$ assumed to be the output of the system (2). The functions a, b and σ can be considered as unknown, as well the system dimension p. It is assumed the relative degree [27] of (2) with respect to σ is $r_\sigma \in \mathbb{N}$, that is

$$\sigma^{(r_\sigma)} = h(t,\phi) + g(t,\phi)u \tag{3}$$

where $h(t,\phi) = \sigma^{(r_\sigma)}|_{u=0}$ and $g(t,\phi) = \frac{\partial \sigma^{(r_\sigma)}}{\partial u}$. Those functions are considered to globally fulfill the inequalities

$$0 < K_m \leq \frac{\partial \sigma^{(r_\sigma)}}{\partial u} \leq K_M \text{ and } |\sigma^{(r_\sigma)}|_{u=0} \leq C$$

for some positive constants K_m, K_M and C, following to the differential inclusion

$$\sigma^{(r_\sigma)} \in [-C,C] + [K_m,K_M]u. \tag{4}$$

The quasi-continuous (QC) homogeneous controller, introduced in [25], provides a bounded control feedback which establishes a finite time sliding mode on the manifold $\sigma = \dot{\sigma} = \ldots \sigma^{(r_\sigma - 1)} = 0$. The control signal is continuous everywhere but in this manifold, so it is called quasi-continuous. For $r_\sigma > 1$, the following functions are defined:

$$\begin{aligned}\varphi_{0,r_\sigma} &= \sigma,\ N_{0,r_\sigma} = |\sigma|,\ \Psi_{0,r_\sigma} = \frac{\varphi_{0,r_\sigma}}{N_{0,r_\sigma}}\\ \varphi_{i,r_\sigma} &= \sigma^{(r_\sigma)} + \beta_i N_{i-1,r_\sigma}^{(r_\sigma - i)/(r_\sigma - i+1)} \Psi_{i-1,r_\sigma}\\ N_{i,r_\sigma} &= |\sigma^{(r_\sigma)}| + \beta_i N_{i-1,r_\sigma}^{(r_\sigma - i)/(r_\sigma - i+1)}\\ \Psi_{i,r_\sigma} &= \frac{\varphi_{i,r_\sigma}}{N_{i,r_\sigma}},\ i = 1,\ldots,r_\sigma.\end{aligned} \tag{5}$$

With the parameters $\beta_1,\ldots,\beta_{r_\sigma},\alpha > 0$ large enough, the controller

$$u = -\alpha \Psi_{r_\sigma - 1, r_\sigma}\left(\dot{\sigma},\ldots,\sigma^{(r_\sigma - 1)}\right) \tag{6}$$

results in a r_σ-sliding homogeneous controller, providing finite tine stability of (4), then inducing a r_σ-sliding mode on (2). The solutions of the closed loop systems are understood in Filippov sense [28].

B. Integral Nested Structure

Block 1: For the first block, let $e_1 = x_1 - y_d$, then

$$\dot{e}_1 = f_1(x_1,t) + B_1(x_1,t)x_2 + \bar{\Delta}_1(x_1,t) \tag{7}$$

where $\bar{\Delta}_1(x_1,t) = \Delta_1(x_1,t) - \dot{y}_d$.

Defining

$$e_2 = x_2 - \phi_1(x_1,t,u_{10},u_{11}) \tag{8}$$

with $\phi_1(x_1,t,u_{10},u_{11}) = B_1^+(x_1,t)[-f_1(x_1,t) + u_{10} + u_{11}]$ and $B_1^+(x_1,t) = B_1^T(x_1,t)\left(B_1^T(x_1,t)B_1(x_1,t)\right)^{-1}$.

Replacing (8) in (7), it follows

$$\dot{e}_1 = u_{10} + u_{11} + B_1(x_1,t)e_2 + \bar{\Delta}_1(x_1,t) \tag{9}$$

where, the control variable u_{10} is used to stabilize the tracking error e_1 and u_{11} is designed such that the disturbance $\bar{\Delta}_1(x_1,t)$ is compensated.

In order to propose the control term u_{11}, the variable $\sigma_1 \in \mathbb{R}^{n_1}$ is defined as

$$\sigma_1 = e_1 + z_1 \tag{10}$$

where z_1 is an integral SM variable, thus

$$\dot{\sigma}_1 = u_{10} + u_{11} + B_1(x_1,t)e_2 + \bar{\Delta}_1(x_1,t) + \dot{z}_1. \tag{11}$$

By selecting dynamics for z_1 as $\dot{z}_1 = -u_{10} - B_1(x_1,t)e_2$, the equation (11) reduces to

$$\dot{\sigma}_1 = u_{11} + \bar{\Delta}_1(x_1,t). \tag{12}$$

The control term u_{10} is proposed as

$$u_{10} = A_1 e_1 \tag{13}$$

with $A_1 \in \mathbb{R}^{n_1 \times n_1}$ being a Hurwitz matrix and, u_{11} selected from (5) as

$$u_{11}^{(r-1)} = \Big[\ -\alpha_{1,1}\Psi_{r-1,r}\left(\dot{\sigma}_{11},\ldots,\sigma_{11}^{(r-1)}\right)\ \ \ldots \\ -\alpha_{1,n_1}\Psi_{r-1,r}\left(\dot{\sigma}_{1n_1},\ldots,\sigma_{1n_1}^{(r-1)}\right)\ \Big]^T \tag{14}$$

where the σ_{1k}, with $k = 1,\ldots,n_1$, is the k-th element of the vector σ_1, and its time derivatives are calculated by using a finite time robust differentiator [29].

Finally, (9) takes the form

$$\dot{e}_1 = A_1 e_1 + u_{11} + B_1(x_1,t)e_2 + \bar{\Delta}_1(x_1,t) \tag{15}$$

with u_{11} as in (14).

The exposed procedure for the *Block 1* will be extended to the blocks i with $i = 2,\ldots,r-1$, as follows:

Block i: For the block i, let $e_i = x_i - \phi_{i-1}$, then

$$\dot{e}_i = f_i(\bar{x}_i,t) + B_i(\bar{x}_i,t)x_{i+1} + \bar{\Delta}_i(\bar{x}_i,t) \tag{16}$$

where $\bar{\Delta}_i(\bar{x}_i,t) = \Delta_i(\bar{x}_i,t) - \dot{\phi}_{i-1}$.

Defining

$$e_{i+1} = x_{i+1} - \phi_i(x_i,t,u_{i0},u_{i1}) \tag{17}$$

with $\phi_i(x_i,t,u_{i0},u_{i1}) = B_i^{+}(\bar{x}_i,t)[-f_i(\bar{x}_i,t) + u_{i0} + u_{i1}]$ and $B_i^{+}(\bar{x}_1,t) = B_i^T(\bar{x}_i,t)\left(B_i^T(\bar{x}_1,t)B_i(\bar{x}_i,t)\right)^{-1}$.

Replacing (17) in (16), it follows

$$\dot{e}_i = u_{i0} + u_{i1} + B_i(\bar{x}_i,t)e_{i+1} + \bar{\Delta}_i(\bar{x}_i,t) \tag{18}$$

where, as for the first block, the control variable u_{i0} is used to stabilize the error variable e_i and u_{i1} is designed such that the disturbance $\bar{\Delta}_i(\bar{x}_i,t)$ is compensated.

To propose the control term u_{i1}, the variable $\sigma_i \in \mathbb{R}^{n_i}$ is defined as

$$\sigma_i = e_i + z_i \tag{19}$$

where z_i is an integral SM variable, thus

$$\dot{\sigma}_i = u_{i0} + u_{i1} + B_i(\bar{x}_i,t)e_{i+1} + \bar{\Delta}_i(\bar{x}_i,t) + \dot{z}_i. \tag{20}$$

With $\dot{z}_i = -u_{i0} - B_i(\bar{x}_i,t)e_{i+1}$, the equation (20) reduces to

$$\dot{\sigma}_i = u_{i1} + \bar{\Delta}_i(\bar{x}_i,t). \tag{21}$$

Similarly, u_{i0} is proposed as

$$u_{i0} = A_i e_i \tag{22}$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$ being a Hurwitz matrix and, u_{i1} selected from (5) as

$$u_{i1}^{(r-i)} = \Big[\ -\alpha_{i,1}\Psi_{r-i,r-i+1}\left(\dot{\sigma}_{i1},\ldots,\sigma_{i1}^{(r-i)}\right)\ \ \ldots \\ -\alpha_{i,n_i}\Psi_{r-i,r-i+1}\left(\dot{\sigma}_{in_i},\ldots,\sigma_{in_i}^{(r-i)}\right)\ \Big]^T \tag{23}$$

where the σ_{ik}, with $k = 1,\ldots,n_i$, is the k-th element of the vector σ_i.

Therefore, (18) takes the form

$$\dot{e}_i = A_i e_i + u_{i1} + B_i(\bar{x}_i,t)e_{i+1} + \bar{\Delta}_i(\bar{x}_i,t) \tag{24}$$

with u_{i1} as in (23).

Finally, the control signal design for the *Block r* will be presented.

Block r: For the block r, let $e_r = x_r - \phi_{r-1}$, then

$$\dot{e}_r = f_r(x,t) + B_r(r,t)u + \bar{\Delta}_r(x,t) \tag{25}$$

where $\bar{\Delta}_r(r,t) = \Delta_r(x,t) - \dot{\phi}_{r-1}$.

Defining

$$u = B_r^{+}(x,t)[-f_r(x,t) + u_{r0} + u_{r1}] \tag{26}$$

with $B_r^{+}(x,t) = B_i^T(x,t)\left(B_r^T(x,t)B_r(x,t)\right)^{-1}$, and replacing (26) in (25), it follows:

$$\dot{e}_r = u_{r0} + u_{r1} + \bar{\Delta}_r(x,t) \tag{27}$$

where, as for the previous blocks, the control variable u_{r0} is used to stabilize the error variable e_r and u_{r1} is designed such that the disturbance $\bar{\Delta}_r(x,t)$ is compensated.

For the control term u_{r1} definition, the variable $\sigma_r \in \mathbb{R}^{n_r}$ is defined as

$$\sigma_r = e_r + z_r \tag{28}$$

where z_r is an integral SM variable, yielding to

$$\dot{\sigma}_r = u_{r0} + u_{r1} + \bar{\Delta}_r(r,t) + \dot{z}_r. \tag{29}$$

With $\dot{z}_r = -u_{r0}$, the equation (29) reduces to

$$\dot{\sigma}_r = u_{r1} + \bar{\Delta}_r(x,t). \tag{30}$$

With the aim to obtain a continuous control u, u_{r0} is proposed as

$$u_{r0} = A_r e_r \tag{31}$$

with $A_r \in \mathbb{R}^{n_r \times n_r}$ being a Hurwitz matrix.

The control term u_{r1} designed with the use of the super-twisting algorithm [30] as

$$u_{r1} = \big[\ -\lambda_{1,1}v_1(\sigma_{r1}) - \lambda_{2,1}v_2(\sigma_{r1})\ \ \ldots \\ -\lambda_{1,n_r}v_1(\sigma_{rn_r}) - \lambda_{2,n_r}v_2(\sigma_{rn_r})\ \big] \tag{32}$$

where the σ_{rk}, with $k = 1,\ldots,n_r$, is the k-th element of the vector σ_r, $\lambda_{1,k},\lambda_{2,k} > 0$ are tuning parameters and, the functions $v_1(\cdot)$, $v_2(\cdot)$ are selected such that

$$v_1(\cdot) = |\cdot|^{1/2}\text{sign}(\cdot) \\ \dot{v}_2(\cdot) = \text{sign}(\cdot).$$

Therefore, (27) takes the form

$$\dot{e}_r = A_r e_r + u_{r1} + \bar{\Delta}_r(x,t) \tag{33}$$

with u_{r1} as in (32).

C. Stability Analysis

The closed loop system, consisting of the equations (15), (24) and (33), has the form

$$\begin{aligned}
\dot{e}_1 &= A_1 e_1 + u_{11} + B_1(x_1,t)e_2 + \bar{\Delta}_1(x_1,t)\\
\dot{e}_2 &= A_2 e_2 + u_{21} + B_2(\bar{x}_2,t)e_3 + \bar{\Delta}_2(\bar{x}_2,t)\\
&\vdots\\
\dot{e}_i &= A_i e_i + u_{i1} + B_i(\bar{x}_i,t)e_{i+1} + \bar{\Delta}_i(\bar{x}_i,t)\\
&\vdots\\
\dot{e}_r &= A_r e_r + u_{r1} + \bar{\Delta}_r(x,t)
\end{aligned} \tag{34}$$

For the block r, by selecting every gain $\lambda_{1,k} > 0$ and $\lambda_{2,k} > \frac{1}{2}\sup_{x,t} \left\| \frac{d}{dt}\bar{\Delta}_r(x,t) \right\|_1$, the manifold $\sigma_r = \dot{\sigma} = 0$ is reached in finite time [31]. Hence, from (30), the equivalent control value [32] for u_{r1}, $\{u_{r1}\}_{\text{eq}}$, rejects the disturbance $\bar{\Delta}_r(x,t)$.

Similarly, note that, with a suitable controller gains selection for the quasi-continuous controllers, for the block i, $i = 1,\ldots,r-1$ a sliding mode is established on the manifold $\dot{\sigma}_1 = \ldots = \sigma_1^{(r-i)} = 0$ in finite time. Hence

$$\{u_{i1}\}_{\text{eq}} = -\bar{\Delta}_i(x_i,t) \tag{35}$$

that is, the equivalent control value $\{u_{i1}\}_{\text{eq}}$ rejects the disturbance $\bar{\Delta}_i(x_i,t)$.

Therefore, the system (34) reduces to

$$\begin{aligned}
\dot{e}_1 &= A_1 e_1 + B_1(x_1,t)e_2\\
\dot{e}_2 &= A_2 e_2 + B_2(\bar{x}_2,t)e_3\\
&\vdots\\
\dot{e}_i &= A_i e_i + B_i(\bar{x}_i,t)e_{i+1}\\
&\vdots\\
\dot{e}_r &= A_r e_r
\end{aligned} \tag{36}$$

which is a linear perturbed system with vanishing perturbation.

The system can be written as

$$\dot{e} = Ae + B(x,t)e \tag{37}$$

with $A = \text{blockdiag}[A_1,\ldots,A_r]$, $e = \text{col}[e_1,\ldots,e_{r-1},0_{n_r}]$ and $B(x,t) = \text{col}[B_1(\cdot),\ldots,B_{r-1}(\cdot),0_{n_r\times n_r}]$.

Since $B(x,t)e$ is bounded, it follows that

$$\|B(x,t)\| < \gamma \|e\| \tag{38}$$

with γ and upper bound of $B(x,t)$.

Consider the system (37). Let $Q = Q^T > 0$ and solve the Lyapunov equation [33].

$$PA + A^T P = -Q. \tag{39}$$

The quadratic Lyapunov function $V = e^T P e$

$$\begin{aligned}
\lambda_{\min}(P)\|e\|^2 \le V &\le \lambda_{\max}(P)\|e\|^2\\
\frac{\partial V}{\partial e}Ae = -e^T P e &\le -\lambda_{\min}(Q)\|e\|\\
\left\|\frac{\partial V}{\partial e}\right\| = \left\|2e^T P\right\| \le 2\|P\|\,\|e\| &= 2\lambda_{\max}(P)\|e\|
\end{aligned} \tag{40}$$

The derivative of V along the trajectories of the system satisfies

$$\dot{V} \le -\lambda_{\min}(Q)\|e\|^2 + 2\lambda_{\max}(P)\gamma\|e\|^2 \tag{41}$$

Hence, the system is globally exponentially stable if $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$.

IV. Numerical Simulation Example

In order to expose the performance of the proposed controller, consider the perturbed third order system [26]

$$\begin{aligned}
\dot{x}_1 &= 2\sin(x_1) + 1.5x_2 + \Delta_1(x_1,t)\\
\dot{x}_2 &= 0.8x_1x_2 + x_3 + \Delta_2(\bar{x}_2,t)\\
\dot{x}_3 &= -x_3^2 + 2u + \Delta_3(x,t)\\
\Delta_1(x_1,t) &= 0.2\sin(t) + 0.1x_1 + 0.12\\
\Delta_2(\bar{x}_2,t) &= 0.3\sin(2t) + 0.2x_1 + 0.2x_2 - 0.4\\
\Delta_3(x,t) &= 0.2\sin(2t) + 0.2x_1 + 0.3x_2 + 0.2x_3 + 0.3.
\end{aligned}$$

where Δ_i, $i = 1,2,3$ are disturbances, which are regarded as unknown to the controller .

For this case, the tracking of the reference $y_d = 2\sin(0.15t) + 4\cos(0.1t) - 4$ by x_1 is desired. For that, a IN-HOSM controller is designed. The controller variables are presented as follows:

The sliding manifolds are

$$\begin{aligned}
\sigma_1 &= e_1 + z_1\\
\sigma_2 &= e_2 + z_2\\
\sigma_3 &= e_3 + z_3
\end{aligned}$$

where the error variables are given by

$$\begin{aligned}
e_1 &= x_1 - y_d\\
e_2 &= x_2 - \phi_1\\
e_3 &= x_3 - \phi_2
\end{aligned}$$

and the integral sliding mode variables are

$$\begin{aligned}
\dot{z}_1 &= -u_{10} - 1.5e_2\\
\dot{z}_2 &= -u_{20} - e_3\\
\dot{z}_3 &= -2u_{30}.
\end{aligned}$$

The quasi-controllers are given by

$$\begin{aligned}
\phi_1 &= \frac{1}{1.5}\left(-2\sin(x_1) + u_{10} + u_{11}\right)\\
\phi_2 &= -0.8x_1x_2 + u_{20} + u_{21}
\end{aligned}$$

which are designed by using the quasi-continuous algorithms for the integral terms

$$\ddot{u}_{11} = -\alpha_1\left(\frac{\ddot{\sigma}_1 + \beta_1\left(|\dot{\sigma}_1| + |\sigma_1|^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(\dot{\sigma}_1 + |\sigma_1|^{\frac{2}{3}}\text{sign}(\sigma_1)\right)}{|\ddot{\sigma}_1| + \beta_1\left(|\dot{\sigma}_1| + |\sigma_1|^{\frac{2}{3}}\right)^{\frac{1}{2}}}\right)$$

$$\dot{u}_{21} = -\alpha_2\left(\frac{\dot{\sigma}_2 + \beta_2|\sigma_2|^{\frac{1}{2}}\text{sign}(\sigma_2)}{|\dot{\sigma}_2| + \beta_2|\sigma_2|^{\frac{1}{2}}}\right)$$

and the nominal terms $u_{10} = -k_1e_1$, $u_{20} = -k_2e_2$.

Similarly, the controller u is given by

$$u = 0.5\left(x_3^2 + u_{30} + u_{31}\right)$$

where $u_{30} = -k_3 e_3$is the nominal control input and the super-twisting $u_{31} = -\lambda_{11} v_1 - \lambda_{21} v_2$, with $v_1 = |\sigma_3|^{\frac{1}{2}} \operatorname{sign}(\sigma_3)$ and $\dot{v}_2 = \operatorname{sign}(\sigma_3)$ is the integral control term.

The closed loop disturbances are $\bar{\Delta}_1(x_1,t) = \Delta_1(x_1,t) - \dot{y}_d$, $\bar{\Delta}_2(\bar{x}_2,t) = \Delta_2(\bar{x}_2,t) - \dot{\phi}_1$ and $\bar{\Delta}_3(x,t) = \Delta_3(x,t) - \dot{\phi}_2$.

The derivatives of the manifolds are obtained with a sliding mode differentiator [29]. The gains for the controller are $k_1 = 5$, $k_2 = 5$, $k_3 = 5$, $\lambda_{11} = 13$, $\lambda_{21} = 15$, $\alpha_1 = 15$, $\alpha_2 = 5$, $\beta_1 = 2$ and $\beta_2 = 1$.

The results obtained by simulation are shown in the following figures. The reference tracking is shown Fig. 1.

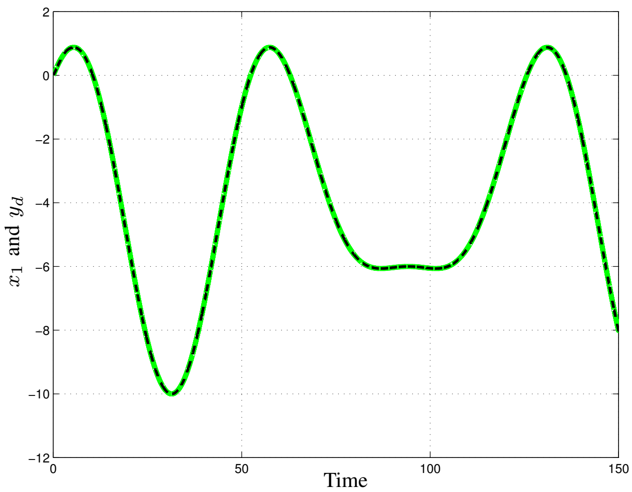

Fig. 1. Reference tracking x_1 (solid) and y_d (dashed)

The control signal is presented in Fig. 2.

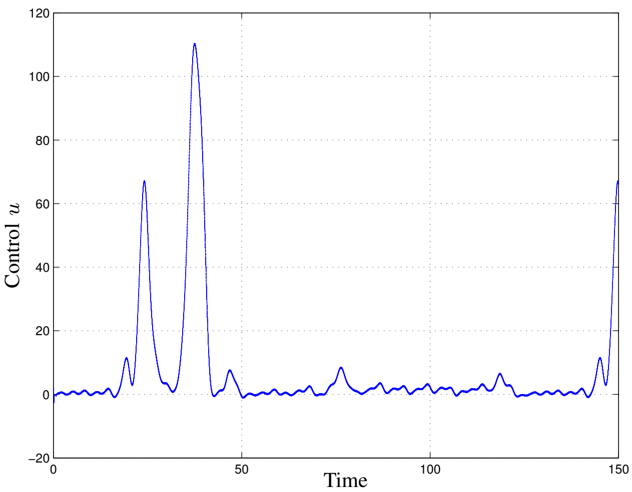

Fig. 2. Control signal u

The simulation exposes the high performance of the controller in presence of, both, matched and unmatched disturbances.

The following figures show how the integral terms of the control u_{11}, u_{21} and u_{31} reject the disturbances $\bar{\Delta}_i(\bar{x}_i,t)$ with $i = 1,2,3$, respectively.

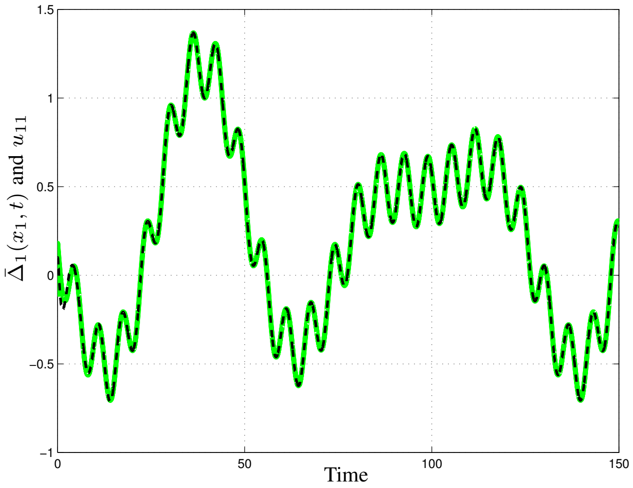

Fig. 3. Disturbance $\bar{\Delta}_1(x_1,t)$ (solid) and $-u_{11}$ (dashed)

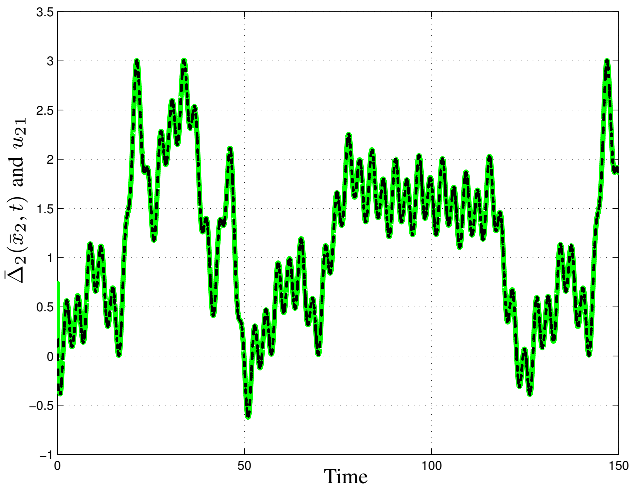

Fig. 4. Disturbance $\bar{\Delta}_2(\bar{x}_2,t)$ (solid) and $-u_{21}$ (dashed)

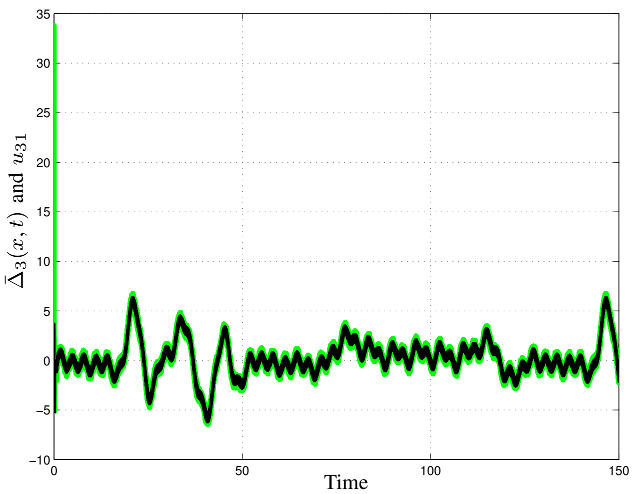

Fig. 5. Disturbance $\bar{\Delta}_3(x,t)$ (solid) and $-u_{31}$ (dashed)

V. CONCLUSIONS

A robust controller for nonlinear systems in the NBC form was presented. This proposal offers finite time exact rejection of, both, matched and unmatched disturbances. Since this robustness features are achieved, an exact exponentially converge tracking is obtained by the closed loop system.

Numerical simulations show the effectiveness and feasibility of the proposal.

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