Multidimensional Digital Signal Estimation Using Kalman´s Theory for Computer-Aided Applications

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ABSTRACT

In this paper, we analyze the Multidimensional Kalman Algorithm to estimate a signal corrupted by white Gaussian noise. Because the theory provide a good solution to the problem with a large number of signals, we developed an algorithm for three-dimensional Kalman filtering applied to the positioning problem (latitude, altitude and longitude) of a stationary object based on GPS signals. This application was selected because the incoming signals of the GPS encounters some noise on its way to the receiver, which is originated from different types of sources, the consequences are that the received signals are noisy, therefore inaccurate. The signals are digitally processed, and the implementation may be carried out on a computer-aided system for a specific application.

Keywords: Signal processing, Kalman algorithm, Estimation theory, Global Positioning System, Digital filtering.

1. INTRODUCTION

The most efficient statistical way to provide optimal linear estimation (filtering, smoothing and prediction) of stationary and non-stationary random signals was proposed by Rudolf E. Kalman, its paper described a recursive solution to the discrete-data linear filtering problem and was published in 1960 [1], and is known now as the Optimal Linear Filtering Theory [2].

The Kalman filter is an estimator for what is called the linear-quadratic problem [3], which is the problem of estimating the instantaneous state of a linear dynamic system perturbed by white noise. The resulting estimator is statistically optimal with respect to any quadratic function of estimation error. The Kalman filter provides means for inferring the missing information from indirect (and noisy) measurements [3]. Scalar case is not widely used in Kalman filtering. The reason is that the filter is not matched with the signal system state model, thus, the filter is not optimal from this point of view. The more general case is the filter state-space equation consists of the proper states number. This is the vector or multi-dimensional case of the Kalman filter [4].

Advances in digital computer technology made possible to consider the implementation of Kalman's recursive solution in a number of real-time applications. The markovian theory of Gaussian processes is a background for the Kalman approach based on which one can estimate a signal through a noisy observation in an optimal way. Nowadays, the Kalman theory is applied not only for the linear signals but also for the non-linear and adaptive problems. In this regards, the Kalman's approach seems like the most universal for the optimal filtering.

The applications of Kalman filtering encompass many fields, but its use as a tool is almost exclusively for two purposes: ***estimation*** and ***performance analysis*** of estimators [3]. The Kalman filter allows us to estimate the state of dynamic systems with certain types of random behavior by using such statistical information. The Kalman filter uses a complete description of the probability distributions of its estimation errors in determining the optimal filter gains, and this probability distributions may be used in assessing its performance as a function of the "design parameters" of an estimation system.

Advantages

Some of the advantages of the Kalman theory are:

1. Is implementable in the form of an algorithm for a digital computer, which was replacing analog circuitry for estimation and control at the time that the Kalman filter was introduced. This implementation may be slower, but it is capable of much greater accuracy than had been achievable with analog filters.

2. Does not require that the deterministic dynamics or the random processes have stationary properties, and many applications of importance include non-stationary stochastic processes.

2. MULTIDIMENSIONAL DISCRETE TIME KALMAN FILTER

Consider a stationary random signal as shown in Figure 1. Such a signal $\xi(t)$, called ***Observation***, is formed by a stationary ***Signal*** $\lambda(t)$ mixed with ***Noise*** $n(t)$. The filtering task then is formulated in the following way: provide the most accurate estimate $\hat{\lambda}(t)$ of a signal $\lambda(t)$ through an observation $\xi(t)$, taking into account that the noise corrupts this estimate, so it is not possible, in principle, to obtain the result with zero error. Once a filter (or a filtering algorithm) provides for the most accurate result then they call it an ***Optimal Filter***. Design of such a filter is the major task of an ***Optimal Filtering***. We have, in principle, two filtering realizations [5]:

a. ***Linear filtering***.- Corresponds to the case of an observation $\xi(t)$ linearly depends on a signal $\lambda(t)$, and a start value λ_0 is normally distributed. For this case, all the processes may be treated as the Gaussian processes.

b. ***Non-linear filtering***.- Corresponds to the case of both/either an observation $\xi(t)$ and/or a signal $\lambda(t)$ are non-linear functions, and a start level λ_0 is a non-Gaussian process.

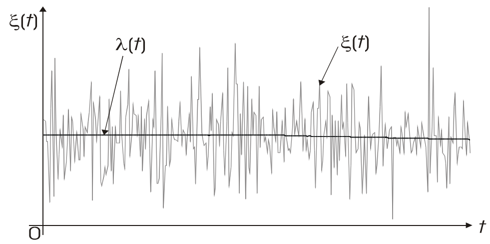

Fig. 1. Stationary Random Signal

Multidimensional Kalman equations

The observation (measurement) of the process ξ_v to be occurred at discrete points in time and the random process λ_v to be estimated can be modeled in the form [5]:

$$\xi_v = \mathbf{H}_v \lambda_v + \mathbf{u}_v + \mathbf{n}_{ov} \tag{1}$$

$$\lambda_v = \mathbf{A}_{v-1}\lambda_{v-1} + \mathbf{n}_{\lambda v} \tag{2}$$

where:

λ_v → ***Signal*** vector, of dimension of (n×1), equals to:

$$\lambda_v = \begin{bmatrix} \lambda_{1v} & \lambda_{2v} & \dots & \lambda_{nv} \end{bmatrix}^T$$

ξ_v → ***Observation*** vector, of dimension of (m×1):

$$\xi_v = \begin{bmatrix} \xi_{1v} & \xi_{2v} & \dots & \xi_{mv} \end{bmatrix}^T$$

$\mathbf{H}_v$ → ***Measurement matrix*** giving the ideal (noiseless) connection between the measurements and the state vector at the time t_v, of dimension of (m×n):

$$\mathbf{H}_v = \begin{bmatrix} H_{11v} & H_{12v} & \dots & H_{1mv} \\ H_{21v} & H_{22v} & \dots & H_{2mv} \\ \dots & \dots & \dots & \dots \\ H_{n1v} & H_{n2v} & \dots & H_{nmv} \end{bmatrix}$$

$\mathbf{u}_v$ → Vector of a ***Control signal***, which equals zero for the purely filtering task, of dimension of (m×1):

$$\mathbf{u}_v = \begin{bmatrix} u_{1v} & u_{2v} & \dots & u_{mv} \end{bmatrix}^T$$

$\mathbf{A}_{v-1}$ → Matrix relating λ_v to λ_{v-1} in the absence of a forcing function (if λ_v is a sample of continuous process, $\mathbf{A}_{v-1}$ is the ***State Transition matrix***), of dimension of (n×n), equals to:

$$\mathbf{A}_v = \begin{bmatrix} A_{11v} & A_{12v} & \dots & A_{1nv} \\ A_{21v} & A_{22v} & \dots & A_{2nv} \\ \dots & \dots & \dots & \dots \\ A_{n1v} & A_{n2v} & \dots & A_{nnv} \end{bmatrix}$$

$\mathbf{n}_{ov}$ → Discrete ***white Gaussian vector noise of an observation***, with mean zero and covariance matrix $\mathbf{V}_v$ of (m×m) dimensions, equals to:

$$E\left[\mathbf{n}_{ov}\mathbf{n}_{ok}^T\right] = \begin{cases} \mathbf{V}_v & , \quad k = v \\ 0 & , \quad k \neq v \end{cases}$$

$\mathbf{n}_{\lambda v}$ → Discrete ***white Gaussian vector noise of a signal***, with mean zero and covariance matrix $\mathbf{\Psi}_v$ of (n×n) dimensions, equals to:

$$E\left[\mathbf{n}_{\lambda v}\mathbf{n}_{\lambda k}^T\right] = \begin{cases} \Psi_v & , \quad k = v \\ 0 & , \quad k \neq v \end{cases}$$

$\mathbf{n}_{ov}$ and $\mathbf{n}_{\lambda v}$ are jointly independent (uncorrelated). So that for all k and v the noises are uncorrelated, this is:

$$E\left[\mathbf{n}_{ov}\mathbf{n}_{\lambda k}^T\right] = 0$$

We assume at this point that we have initial estimate of the process at some points in time t_v, and that this estimate is based on all our knowledge about the process prior to t_v. This prior estimate will be denoted as $\hat{\lambda}_{v-1}$, where the "hat" denotes estimate, and the negative sign "-1" is a reminder that this is our best estimate prior to assimilating the measurement at t_v. We also assume that we know the error covariance matrix associated with $\hat{\lambda}_{v-1}$. That is, we denote the estimation error:

$$\varepsilon_{v-1} = \lambda_v - \mathbf{A}_{v-1}\hat{\lambda}_{v-1} \tag{3}$$

And the associated error covariance matrix is for the estimation error of mean-zero:

$$\mathbf{R}_{v-1} = E\left[\varepsilon_{v-1}\varepsilon_{v-1}^T\right] = E\left[\left(\lambda_v - \mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right)\left(\lambda_v - \mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right)^T\right] \tag{4}$$

In many cases, we begin the estimation problem with no prior measurements. Thus, in this case, if the process is zero, the initial estimate is zero, and the associated error covariance matrix is just the covariance matrix of λ itself. With the assumption of a priori estimate $\hat{\lambda}_{v-1}$, we now seek to use the measurement ξ_v to improve the prior estimate. We choose a linear blending of the noisy measurement and the prior estimate in accordance with the equation:

$$\hat{\lambda}_v = \mathbf{A}_{v-1}\hat{\lambda}_{v-1} + \mathbf{K}_v\left(\xi_v - \mathbf{u}_v - \mathbf{H}_v\mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right) \tag{5}$$

where $\hat{\lambda}_v$ is updated estimate, $\mathbf{K}_v$ is filter gain (blending factor), which yet to be determined.

The Kalman Gain

The problem now is to find the filter gain $\mathbf{K}$, that yields an updated estimate that is optimal in some sense. We use minimum mean-square error as the performance criterion [5]. Towards this end, we first form the expression for the error covariance matrix associated with the updated (a posteriori) estimate:

$$\mathbf{R}_v = E\left[\varepsilon_v\varepsilon_v^T\right] = E\left[\left(\lambda_v - \hat{\lambda}_v\right)\left(\lambda_v - \hat{\lambda}_v\right)^T\right] \tag{6}$$

Next, we substitute Eq. (1) into Eq. (5) and then substitute the resulting expression for $\hat{\lambda}_v$ into Eq. (6). The result is:

$$\mathbf{R}_v = E\left\{\left[\left(\lambda_v - \mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right) - \mathbf{K}_v\left(\mathbf{H}_v\lambda_v + \mathbf{n}_{ov} - \mathbf{H}_v\mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right)\right] \times \left[\left(\lambda_v - \mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right) - \mathbf{K}_v\left(\mathbf{H}_v\lambda_v + \mathbf{n}_{ov} - \mathbf{H}_v\mathbf{A}_{v-1}\hat{\lambda}_{v-1}\right)\right]^T\right\} \tag{7}$$

Now, performing the indicated expectation and nothing the $\left(\lambda_v - \hat{\lambda}_{v-1}\right)$ is the a priori estimation error that is uncorrelated with the measurement error $\mathbf{n}_{ov}$, we have:

$$\mathbf{R}_v = \left(\mathbf{I} - \mathbf{K}_v\mathbf{H}_v\right)\mathbf{R}_{v-1}\left(\mathbf{I} - \mathbf{K}_v\mathbf{H}_v\right)^T + \mathbf{K}_v\mathbf{V}_v\mathbf{K}_v^T \tag{8}$$

where $\mathbf{I}$ is unit matrix. Notice that Eq. (8) is a perfectly general expression for the updated error covariance matrix, and it applies for any gain $\mathbf{K}_v$, suboptimal or otherwise. Returning to the optimization problem, we wish to find the particular $\mathbf{K}_v$ that minimizes the individual terms along the major diagonal of $\mathbf{R}_v$ Eq. (8), because these terms represents the estimation error variances of the elements of the state vector λ_v being estimated. The optimization can be done in a number of ways. We will try to do using a straightforward differential calculus approach, and to do so we need two matrix differential formulas. They are:

$$\frac{d\left[\mathrm{trace}\left(\mathbf{AB}\right)\right]}{d\mathbf{A}} = \mathbf{B}^T \quad , \quad \left(\mathbf{AB} \text{ must be square}\right) \tag{9}$$

$$\frac{d\left[\mathrm{trace}\left(\mathbf{ACA}^T\right)\right]}{d\mathbf{A}} = 2\mathbf{AC} \quad , \quad \left(\mathbf{C} \text{ must be symmetric}\right) \tag{10}$$

where the derivative of a scalar with respect to a matrix is:

$$\frac{ds}{d\mathbf{A}} = \begin{bmatrix} \frac{ds}{da_{11}} & \frac{ds}{da_{12}} & \dots \\ \frac{ds}{da_{21}} & \frac{ds}{da_{22}} & \dots \\ \dots & \dots & \dots \end{bmatrix} \tag{11}$$

We will now expand the general form for $\mathbf{R}_\nu$ on Eq. (8), and rewrite it in the form:

$$\mathbf{R}_\nu = \mathbf{R}_{\nu-1} - \mathbf{K}_\nu \mathbf{H}_\nu \mathbf{R}_{\nu-1} - \mathbf{R}_{\nu-1}\mathbf{H}^{\mathrm{T}}\mathbf{K}_\nu^{\mathrm{T}} + \\ +\mathbf{K}_\nu\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)\mathbf{K}_\nu^{\mathrm{T}} \quad (12)$$

The second and third terms are linear in $\mathbf{K}_\nu$ and that the fourth term is quadratic in $\mathbf{K}_\nu$. The two matrix differentiation formulas may now be applied to Eq. (12). We wish to minimize the trace of $\mathbf{R}_\nu$ because it is the sum of the mean-square errors in the estimates of all the elements of the state vector. We can use the argument here that the individual mean-square errors are also minimized when the total is minimized, provided that we have enough degrees of freedom in the variation of $\mathbf{K}_\nu$, which we do in this case. We proceed now to differentiate the trace of $\mathbf{R}_\nu$, with respect to $\mathbf{K}_\nu$, and we note that the trace of $\mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}}\mathbf{K}_\nu^{\mathrm{T}}$ is equal to the trace of its transpose $\mathbf{K}_\nu\mathbf{H}_\nu\mathbf{R}_{\nu-1}$. The result is:

$$\frac{d\left(\text{trace } \mathbf{R}_\nu\right)}{d\mathbf{K}_\nu} = -2\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\right)^{\mathrm{T}} + 2\mathbf{K}_\nu\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right) \quad (13)$$

We now set the derivative equal to zero and solve for the optimal gain. The result is:

$$\mathbf{K}_\nu = \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}}\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)^{-1} \quad (14)$$

where $\mathbf{R}_{\nu-1}^{\mathrm{T}} = \mathbf{R}_{\nu-1}$. This particular $\mathbf{K}_\nu$, namely, the one that minimizes the mean-square estimation error, is called the ***Kalman Gain***.

The Kalman Filter Error

The covariance matrix associated with the optimal estimate may now be computed by comparing Eq. (8) and Eq. (12). We the put equality and have:

$$\mathbf{R}_\nu = \left(\mathbf{I} - \mathbf{K}_\nu \mathbf{H}_\nu\right)\mathbf{R}_{\nu-1}\left(\mathbf{I} - \mathbf{K}_\nu \mathbf{H}_\nu\right)^{\mathrm{T}} + \mathbf{K}_\nu \mathbf{V}_\nu \mathbf{K}_\nu^{\mathrm{T}} \\ = \mathbf{R}_{\nu-1} - \mathbf{K}_\nu \mathbf{H}_\nu \mathbf{R}_{\nu-1} - \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}}\mathbf{K}_\nu^{\mathrm{T}} + \mathbf{K}_\nu\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)\mathbf{K}_\nu^{\mathrm{T}} \quad (15)$$

Routine substitution of the optimal Kalman gain expression of Eq. (14) into Eq. (15) leads to [5]:

$$\mathbf{R}_\nu = \mathbf{R}_{\nu-1} - \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}}\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)^{-1}\mathbf{H}_\nu \mathbf{R}_{\nu-1} \quad (16)$$

or

$$\mathbf{R}_\nu = \mathbf{R}_{\nu-1} - \mathbf{K}_\nu\left(\mathbf{H}_\nu \mathbf{R}_{\nu-1}\mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)^{-1}\mathbf{K}_\nu^{\mathrm{T}} \quad (17)$$

or

$$\mathbf{R}_\nu = \left(\mathbf{I} - \mathbf{K}_\nu \mathbf{H}_\nu\right)\mathbf{R}_{\nu-1} \quad (18)$$

We have four expressions for computing the updated $\mathbf{R}_\nu$ from the priori $\mathbf{R}_{\nu-1}$. Three of these, Eq. (16), Eq. (17) and Eq. (18), are only valid for the optimal gain condition. However, Eq. (8) is valid for any gain, optimal or suboptimal. All four equations yield identical results for optimal gain with perfect arithmetic. We note, though, that in the real engineering world Kalman filtering is a numerical procedure, and some of the **R**-update equations may perform better numerically than others under unusual conditions.

The Kalman Filter Algorithms

We now would like to generalize the above consideration while presenting the Kalman optimal filtering algorithms for two common situations [5].

The first case of $\mathbf{m} \geq \mathbf{n}$: This case corresponds to the situation when the number of observations is more or equal to the number of the states. First we predict the initial error $\mathbf{R}_{\nu-1}$ and estimate $\hat{\lambda}_{\nu-1}$. The Kalman optimal algorithm then becomes:

$$\mathbf{R}_\nu^{-1} = \left[\mathbf{A}_{\nu-1}^{\mathrm{T}}\mathbf{R}_{\nu-1}\mathbf{A}_{\nu-1} + \Psi_\nu\right]^{-1} + \mathbf{H}_\nu^{\mathrm{T}}\mathbf{V}_\nu^{-1}\mathbf{H}_\nu \quad (19)$$

$$\mathbf{K}_\nu = \mathbf{R}_\nu \mathbf{H}_\nu^{\mathrm{T}}\mathbf{V}_\nu^{-1} \quad (20)$$

$$\hat{\lambda}_\nu = \mathbf{A}_{\nu-1}\hat{\lambda}_{\nu-1} + \mathbf{K}_\nu\left(\xi_\nu - \mathbf{u}_\nu - \mathbf{H}_\nu \mathbf{A}_{\nu-1}\hat{\lambda}_{\nu-1}\right) \quad (21)$$

$\nu = \nu + 1$

where the gain $\mathbf{K}_\nu$ on Eq. (20) is written straightforward based on the one-dimensional case. After the first circle, we change step $\nu=\nu+1$ and update estimates.

The second case of m<n: We have considered this case above and this is the situation when the number of observations is less than the number of the states. The alternative way to transfer from the Eq. (19), Eq. (20) and Eq. (21) to those equations is based on the use of the *lemma of matrix conversion*, which yields:

$$\left(\mathbf{R}_{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{H}\right)^{-1} = \mathbf{R} - \mathbf{R}\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{R}\mathbf{H}^{\mathrm{T}} + \mathbf{N}\right)^{-1}\mathbf{H}\mathbf{R}$$

The Kalman optimal algorithm for assumed $\mathbf{R}_{\nu-1}$ and $\hat{\lambda}_{\nu-1}$ becomes then as follows:

$$\tilde{\mathbf{R}}_\nu = \mathbf{A}_{\nu-1}^{\mathrm{T}}\mathbf{R}_{\nu-1}\mathbf{A}_{\nu-1} + \Psi_\nu \quad (22)$$

$$\mathbf{K}_\nu = \tilde{\mathbf{R}}_\nu \mathbf{H}_\nu^{\mathrm{T}}\left(\mathbf{H}_\nu \tilde{\mathbf{R}}_\nu \mathbf{H}_\nu^{\mathrm{T}} + \mathbf{V}_\nu\right)^{-1} \quad (23)$$

$$\hat{\lambda}_\nu = \mathbf{A}_{\nu-1}\hat{\lambda}_{\nu-1} + \mathbf{K}_\nu\left(\xi_\nu - \mathbf{u}_\nu - \mathbf{H}_\nu \mathbf{A}_{\nu-1}\hat{\lambda}_{\nu-1}\right) \quad (24)$$

$$\mathbf{R}_\nu = \left(\mathbf{I} - \mathbf{K}_\nu \mathbf{H}_\nu\right)\tilde{\mathbf{R}}_\nu \quad (25)$$

$\nu = \nu + 1$

where sign "~" denotes prediction on one step ahead. Now note that for all the multidimensional Kalman algorithms, an analytic expression of the filter error via error matrix on Eq. (22) and gain on Eq. (25) is problematic requiring an extremely routine way. The computer-aided calculation is more preferable. The Figure 2 shows the Kalman algorithm schematically. It follows that before computing, we must enter initial values of error and estimate, and the follow the recursive procedure. Each step of computation is based on the new measurement data and the result is appeared in a form of the current estimate.

3. MULTIDIMENSIONAL KALMAN FILTER STRUCTURE

The Figure 3 shows the multidimensional structure of the recursive Kalman algorithm.

4. REAL APPLICATION OF THE MULTIDIMENSIONAL KALMAN FILTER

Navigation is defined as the science of getting a craft o person from one place to another [6]. In some cases a more accurate knowledge of either our position, intended course and transit time to a desired destination is required, on this situations navigation aids are used. Some navigation aids are very complex and transmit electronic signals, that are referred to as radionavigation aids. Signals from one or more radionavigation aids enable a person to compute their position.

Various types of radionavigation aids exist. The Global Positioning System (GPS) was created in the early 1960s by the National Aeronautics and Space Administration (NASA), developing satellite systems for positioning determination.

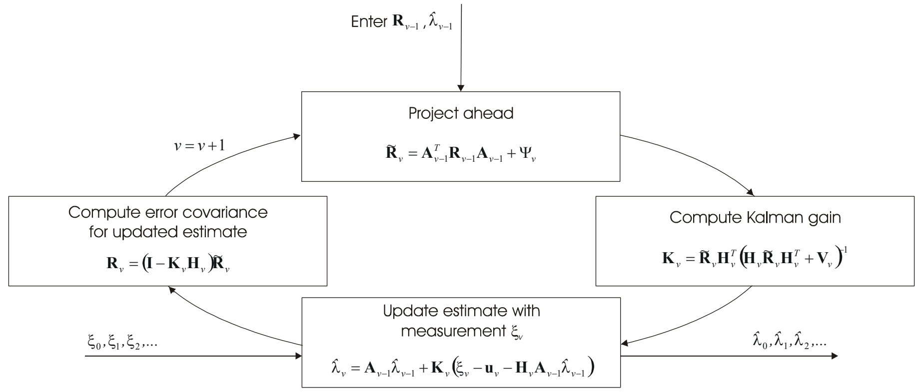

Fig. 2. Kalman Filter Algorithm

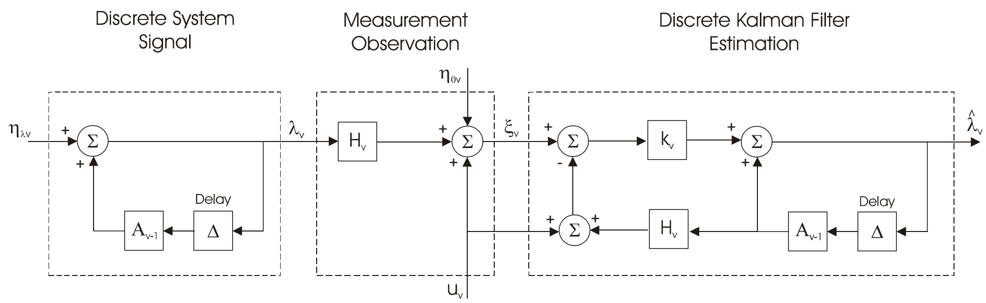

Fig. 3. Multidimensional Kalman Filter structure

The susceptibility of the GPS signals to interference is of concern to the GPS user community. Because of the low receiver power of the GPS signals, outages can easily occur due to unintentional interference. There are two factors that determine the accuracy of a GPS position [7]. The first factor, the error on a measurement, has many components that are controlled by the receiver and the local environment of the system. The second factor, the ranging errors, are grouped into the six following classes: ephemeris data errors, satellite clock errors, ionosphere errors, troposphere errors, multipath errors and receiver errors [8].

The algorithm defined by the figures 1 and 2 is going to be implemented for optimal 3-dimensional Kalman filtering to reduce the noise of the observation signals to obtain optimum estimates those which will be as accurate as possible, based only on stationary signals. This will provide the most accurate position (latitude, altitude and longitude) of a stationary object located anywhere on the Earth, which uses a GPS-based positioning receiver. Because this algorithm will be used to estimate the three-dimensional position of a stationary object, it is necessary to work with three different signals (one for latitude, one for altitude and one for longitude). For the design and simulation process, we will use noisy signals obtained from a UT+ Oncore receiver. The digital information of those signals are stored in files, whose characteristics are [9]:

a. Each one of the input files contains two discrete time signals following with digitization time T=100sec.
b. The first signals ξ_v are obtained as the result of the OCXO ~5Mhz frequency measurement based on GPS reference 1pps timing signals. They are presented as time difference in the units of 100psec to 1sec that corresponds to 10^{-10} of frequency instability.
c. The seconds signals λ_v are the real OCXO frequency instability in time measured with reference to the quantum rubidium standard of frequency. Those signals are presented in the units of 10^{-12}.

The Observations signals ξ_v are received, but those observable signals are noisy. Because this is a design problem, we already know the Signals λ_v (contained on the input files, respectively). The next step will be to compare the estimates with the original signal, to ensure that the filter can provide an optimal estimation with minimum error. Of course, in real practice, the Signals λ_v will not be known, but this approach will show that the filter will work with any type of GPS's noisy signals, providing the best stationary estimation (the signals λ_v from the input files are used only to compare the results, but they don't affect the application of the Kalman filter in any way).

Application of the Algorithm to the Positioning problem

The noisy signals ξ_v received from the GPS consists of the sum of the original signal λ_v and white Gaussian noise n_{ov}. Based on Eq. (1), the three Observation signals can be described as:

$$\xi x_v = \alpha \cdot \lambda x_v + nx_{ov} \quad (26)$$

$$\xi y_v = \beta \cdot \lambda y_v + ny_{ov} \quad (27)$$

$$\xi z_v = \gamma \cdot \lambda z_v + nz_{ov} \quad (28)$$

where:

$\xi x_v, \xi y_v, \xi z_v$ → Are the discrete ***Observation Signals*** for latitude, altitude and longitude, respectively.

$\lambda x_v, \lambda y_v, \lambda z_v$ → Are the discrete ***Signals*** for latitude, altitude and longitude, respectively.

$nx_{ov}, ny_{ov}, nz_{ov}$ → Are the discrete ***White Gaussian noises of the observations*** for latitude, altitude and longitude, respectively.

α, β, γ → Are constant values that gives the ideal (noiseless) connection between the observations (measurements) and the original signals.

Following this, based on Eq. (2), the original signals can be modeled as:

$$\lambda x_v = \delta \cdot \lambda x_{v-1} + nx_{\lambda v} \quad (29)$$

$$\lambda y_v = \varphi \cdot \lambda y_{v-1} + ny_{\lambda v} \quad (30)$$

$$\lambda z_v = \rho \cdot \lambda z_{v-1} + nz_{\lambda v} \quad (31)$$

where:

λx_v, λy_v, λz_v → Are the discrete ***Signals*** for latitude, altitude and longitude, respectively.

λx_{v-1}, λy_{v-1}, λz_{v-1} → Are the discrete a-priori values of the ***signals*** for latitude, altitude and longitude, respectively.

$nx_{\lambda v}$, $ny_{\lambda v}$, $nz_{\lambda v}$ → Are the discrete ***White Gaussian noises of the signals*** for latitude, altitude and longitude, respectively.

δ, φ, ρ → Are constant values that relates λ_v to λ_{v-1} in the absence of a forcing function.

It is assumed that the connection between the observations and the signals is ideal (in the noiseless case), so, the constant α, β and γ values are equal to 1. In the same way, in the noiseless and stationary case, the a priori values λ_{v-1} are the same than λ_v, so, the constant δ, φ and ρ values are equal to 1. We already know that $\mathbf{n}_{ov}$ and $\mathbf{n}_{\lambda v}$ are discrete jointly independent (uncorrelated) white Gaussian vector noises of an observation and signal, respectively. $\mathbf{n}_{ov}$ is a vector with mean-zero and covariance matrix $\mathbf{V}_v$. This matrix is equal to:

$$\mathbf{V}_v = \begin{pmatrix} E\{nx_{ov} nx_{ov}^T\} & E\{nx_{ov} ny_{ov}^T\} & E\{nx_{ov} nz_{ov}^T\} \\ E\{ny_{ov} nx_{ov}^T\} & E\{ny_{ov} ny_{ov}^T\} & E\{ny_{ov} nz_{ov}^T\} \\ E\{nz_{ov} nx_{ov}^T\} & E\{nz_{ov} ny_{ov}^T\} & E\{nz_{ov} nz_{ov}^T\} \end{pmatrix} \quad (32)$$

For independent and uncorrelated observation noises [5]:

$$V_{11v} = E\{nx_{ov} nx_{ov}^T\} = Dx_{ov} = Sx_{ov} \cdot \frac{1}{\Delta} = \frac{Nx_{ov}}{2\Delta} = \sigma_{xov}^2 \quad (33)$$

$$V_{22v} = E\{ny_{ov} ny_{ov}^T\} = Dy_{ov} = Sy_{ov} \cdot \frac{1}{\Delta} = \frac{Ny_{ov}}{2\Delta} = \sigma_{yov}^2 \quad (34)$$

$$V_{33v} = E\{nz_{ov} nz_{ov}^T\} = Dz_{ov} = Sz_{ov} \cdot \frac{1}{\Delta} = \frac{Nz_{ov}}{2\Delta} = \sigma_{zov}^2 \quad (35)$$

$\mathbf{n}_{\lambda v}$ is a vector with mean-zero and covariance matrix $\mathbf{\Psi}_v$. It is supposed that the original signal is not known, so, covariance matrix $\mathbf{\Psi}_v$ is also unknown. But this matrix represents the noise level that we want to obtain on the estimation signals, then the matrix can be adjusted to find the minimum error on the estimates. The noises between estimates must be uncorrelated, and the variance between the same signals can be defined from the matrix $\mathbf{V}_v$, this is:

$$\Psi_v = \begin{pmatrix} \upsilon \cdot \sigma_{xov}^2 & 0 & 0 \\ 0 & \tau \cdot \sigma_{yov}^2 & 0 \\ 0 & 0 & \varsigma \cdot \sigma_{zov}^2 \end{pmatrix} \quad (36)$$

where υ, τ and ζ are constant values that will be adjust to obtain the best estimations. Now, following the algorithm shown on Figure 2 and Figure 3, the calculations can be made. Also, the errors between the estimates and original signals will be determined using the following equation:

$$e = \hat{\lambda}_v - \lambda_v \quad (37)$$

Programming the Optimal Algorithm

To obtain the best estimation of the signals, it is necessary to use a medium that provide accuracy results. The results are carried out in a computer-aided implementation using MathLAB© software. This software provide us accurate and fast results, and can be transferred to another applications as Simulink© software.

5. THE RESULTS

The computer's code implemented in MatLAB© software is executed and the expected results are obtained. The program calculates the estimates based on the conditions previously described and displays the results. The estimates obtained from this algorithm are shown in the Figures 4, 5 and 6 for the signal *x*, *y* and *z* (latitude, altitude and longitude), respectively, showing the observation signals (ξx_v, ξy_v, ξz_v), the original signals (λx_v, λy_v, λz_v) and the estimated signals ($\hat{\lambda} x_v$, $\hat{\lambda} y_v$, $\hat{\lambda} z_v$), where it is possible to see that the estimates are very approximated to the original signals.

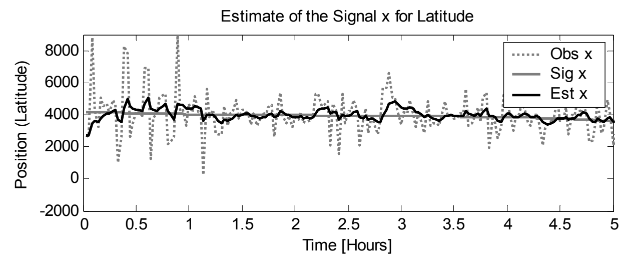

Fig. 4. Kalman estimate of the signal x (Latitude)

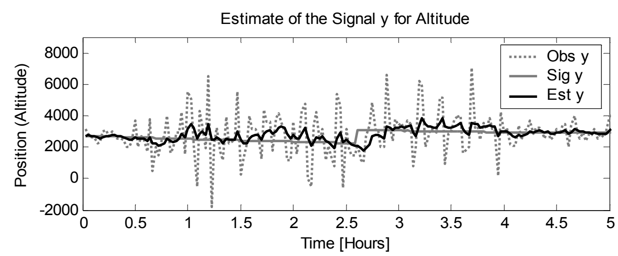

Fig. 5. Kalman estimate of the signal y (Altitude)

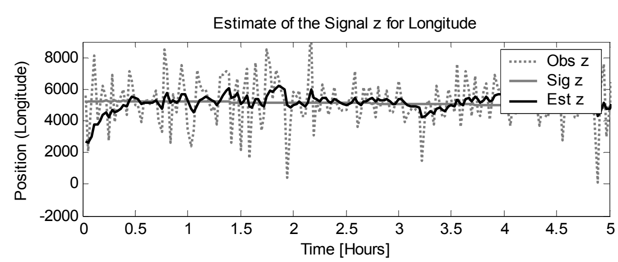

Fig. 6. Kalman estimate of the signal z (Longitude)

The Figure 7 shows the behavior of the error for each estimate, which are described in the form of the Eq. (37). It is clear to see that the mean of the errors are very approximated to zero in each case, this means that the estimates are closest to the original values and follows them. The statistical values for the errors are described in the Table 1. It is clear to see from the results that the 3-Dimensional Kalman Filter produces very accurate results for this particular example.

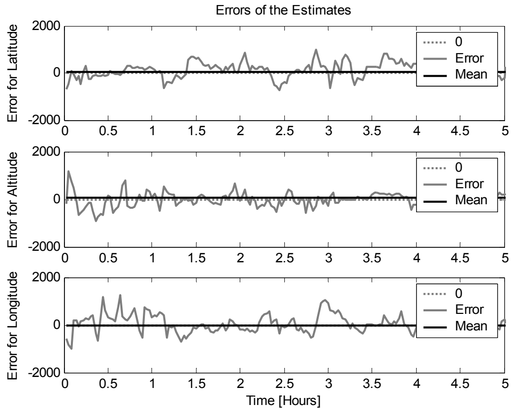

Fig. 7. Behavior of the error function for signals x, y and z

	Signal x	Signal y	Signal z
Mean value of the Original Signal	4026.66	2935.20	5029.75
Mean value of the Estimate	4059.92	3008.72	5017.71
Mean of the error function, ns	31.35	71.77	-14.17
Root Mean Square Deviation, ns	349.35	285.02	469.40
Root Mean Square Error, ns	350.76	293.92	469.61
Maximal value of the Error	1460.30	1152.73	1767.88

Table 1. Statistical values from the 3D Kalman estimates

6. CONCLUSIONS

The Kalman's filter provides a good estimation of a stationary signal. This paper showed the methodology to develop this algorithm for the multidimensional digital signal estimation and its implementation on software for computer-aided applications. This is a very important task, because several signal can be processed using this algorithm in a very convenient and fast way. The positioning estimation problem was used to show the development of the algorithm for a particular application because this issue is becoming very important in our modern life. It is very difficult to imagine an airplane or a ship traveling with the most sophisticated electronics on board but still using a compass or similar instruments to determine their positions. The Kalman filter is not the only available to estimate multidimensional digital signals. However, the digital signal process developed on this paper provides very good results compared with another methods [10].

Advances on electronics and radiocommunications are revolutionizing the way we look our world. Precise position used on airplanes can help pilots to land with zero visibility or avoid ship collisions on a foggy day. All these examples are for non-stationary objects, but the stationary ones play also a critical role, by example, to determine the precise position of a shipwreck or the position of an archeological discovery in the middle of the desert.

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