An Integral Sliding Mode Observer for Linear Systems

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Abstract: In this paper a sliding-mode observer for linear time-invariant systems is proposed. The observer is based on integral sliding modes and the equivalent control method. In order to induce a sliding mode in the output error, a second order sliding mode algorithm is used. Convergence proofs of the proposed observer are presented. In order to expose the features of this proposal, a design example over a DC motor model is exposed, noiseless and noisy measurements cases are considered. For this case, the simulation shows the high performance of the integral observer.

Keywords: Integral Controllers, Linear Systems, Sliding-Mode Control, State Observers.

1. INTRODUCTION

A large amount of controller design methods are developed under the assumption that the state vector is available. However, the state vector can not always be completely measured, but a part of it (Luenberger, 1964). This is due to several reasons, such as there are no on-line sensors for some variables, sometimes it is impossible to install sensors due to hostile environments and some sensors are very expensive or with poor accuracy.

The state observers have taken place as a solution to this issues. The purpose of a state observer is to estimate the unmeasured state variables based on the measured inputs and outputs. Often, an observer is a replica of the original system mathematical model plus a correction signal depending on the difference between the system measured variables and the observer outputs (Luenberger, 1964; Walcott et al., 1987; Kalman, 1960; Kalman and Bucy, 1961).

Several state observers for linear systems have been proposed. A first approach is the Luenberger observer. Here, the observation problem is treated for the case when the system is completely deterministic (no statistical processes are involved) (Luenberger, 1964). When the output measurements are corrupted by zero mean, uncorrelated and white noise, the well-known Kalman Filter provides the optimal solution, once the statistical properties of noise are known (Kalman, 1960; Kalman and Bucy, 1961).

As alternative, an important class of state observers are the sliding mode observers (SMO) which have the main features of the sliding mode (SM) algorithms (Utkin, 1992). Those algorithms, are proposed with the idea to drive the dynamics of a system to an sliding manifold, that is an integral manifold with finite reaching time (Drakunov and Utkin, 1992), exhibiting very interesting features such as work with reduced observation error dynamics, the possibility of obtain a step by step design, robustness and insensitivity under parameter variations and external disturbances, and finite time stability (Utkin, 1992). In addition, some SMO have attractive properties similar to those of the Kalman filter (i.e. noise resilience) but with simpler implementation (Drakunov, 1983). Sometimes this design can be performed by applying the equivalent control method (Drakunov, 1992; Drakunov and Utkin, 1995), allowing the proposal of robust to noise observers, since the equivalent control is slightly affected by noisy measurements. On the other hand, a common and effective approach to sliding mode control is the integral SM (Matthews and DeCarlo, 1988; Utkin and Shi, 1996; Fridman et al., 2006; Galván-Guerra and Fridman, 2013). Here, it is designed an sliding manifold such that the sliding motion has the same dimension that the original system but without the influence of the matched disturbances. Those disturbances belong to the span of the control function and are rejected for the equivalent control obtained from induce the integral SM (Draženović, 1969). In order to propose that manifold, integral SM terms are designed based on the nominal system. When the system initial conditions are known, this control algorithm can be proposed with the aim to force the system trajectory starting from the sliding manifold, eliminating the reaching phase and ensuring robustness from the initial time.

Consequently, in this paper an integral sliding mode-based observer for linear systems is proposed. The observer structure is similar to the observer presented on Drakunov (1992), but using integral SM. In addition, a step by step design of the proposed observer is provided along with a design example over a DC motor model.

The following sections are organized as follows: Section 2 presents the preliminaries for the observer. In Section 3, the integral SM observer is presented. A design example is analyzed in Section 4. In Section 5 the simulation results are shown. Finally, the conclusions of this paper are presented in Section 6.

2. MATHEMATICAL PRELIMINARIES

This section presents the previous results needed for the proposed observer.

2.1 The Super-Twisting Algorithm

Consider the first order perturbed system

$$\dot{\xi} = -u + \Delta, \tag{1}$$

where $\xi, \Delta, u \in \mathbb{R}$.

The super-twisting controller $u = \mathcal{ST}(\xi)$ (Levant, 1993), is defined as

$$\begin{aligned} \mathcal{ST}(\xi) &= \alpha_1 |\xi|^{\frac{1}{2}} \operatorname{sign}(\xi) + w \\ \dot{w} &= \alpha_2 \operatorname{sign}(\xi), \end{aligned} \tag{2}$$

with $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(0) \in \{-1, 1\}$.

For the system (1), the controller (2) is applied, yielding the closed loop system:

$$\begin{aligned} \dot{\xi} &= -\alpha_1 |\xi|^{\frac{1}{2}} \operatorname{sign}(\xi) + q \\ \dot{q} &= -\alpha_2 \operatorname{sign}(\xi) + \dot{\Delta}, \end{aligned} \tag{3}$$

where $q = w + \Delta$.

Assuming that $\left|\dot{\Delta}\right| < \delta$, the super-twisting gains are selected as: $\alpha_1 = 1.5\delta^{\frac{1}{2}}$ and $\alpha_2 = 1.1\delta$. Therefore, a sliding mode is induced on the manifold $(\xi, q) = (0, 0)$ in a finite-time $t_q > 0$ (Moreno and Osorio, 2008). Thus, from (3), the term w in (2) becomes equal to $-\Delta$.

Now, consider the multi-variable case, with $\xi = [\xi_1 \ \dots \ \xi_p]^T, \Delta = [\Delta_1 \ \dots \ \Delta_p]^T, u = [u_1 \ \dots \ u_p]^T \in \mathbb{R}^p$. Assuming $\left\|\dot{\Delta}\right\| < \delta$, it can be shown that $\left|\dot{\Delta}_i\right| < \delta_i \ \forall i \in 1, \dots, p$. In this case, define $u = \mathcal{ST}(\xi) = [\mathcal{ST}(\xi_1) \ \dots \ \mathcal{ST}(\xi_p)]$ and note that this multi-variable case is simply the same as having p (1)-like scalar systems.

2.2 Linear Systems

Consider the following time-invariant linear system represented by the following state space equation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^k$ is the output vector, $A \in \mathbb{R}^{n\times n}$ is the transition matrix, $B \in \mathbb{R}^{n\times m}$ is the input-state distribution matrix and $C \in \mathbb{R}^{k\times n}$ is the output matrix, which will be assumed to have full row rank so the measured outputs are independent. Additionally, it will be assumed that the pair (A, C) is observable.

This paper deals with the case when the measured output is a part of the state. In this case, the system (4) can be rewritten as:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= x_1, \end{aligned} \tag{5}$$

where $A_{11} \in \mathbb{R}^{k\times k}$, $A_{12} \in \mathbb{R}^{k\times(n-k)}$, $A_{21} \in \mathbb{R}^{(n-k)\times k}$, $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, $B_1 \in \mathbb{R}^{k\times m}$, $B_2 \in \mathbb{R}^{(n-k)\times m}$, are partitions of the matrices A and B, such that:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix};$$

$y = x_1 \in \mathbb{R}^k$ is the measured part of the state vector and $x_2 \in \mathbb{R}^{(n-k)}$ is the unmeasured part of the state vector.

Many linear systems can be directly expressed in the form described by (5) (i.e., the measured output is a part of the state vector). If not, under the assumption that C is full rank, there is always a linear transformation which allows to express the system (4) in the form (5), as described in (Utkin, 1992). For instance, assuming the output vector y may be represented as:

$$y = K_1x_1 + K_2x_2, \ x^T = [x_1 \ x_2]^T, \ x_1 \in \mathbb{R}^k, \ x_2 \in \mathbb{R}^{(n-k)},$$

consider a coordinate transformation $x \mapsto Tx$ associated with the invertible matrix

$$T = \begin{bmatrix} K_1 & K_2 \\ I_k & 0 \end{bmatrix}$$

Applying the change of coordinates $x \mapsto Tx$, the triplet (A, B, C) has the form:

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ CT^{-1} = [I_k \ 0].$$

3. INTEGRAL SLIDING MODE OBSERVER

3.1 Observer Scheme

Based on (5), the following state observer is proposed:

$$\begin{aligned} \dot{\hat{x}}_1 &= A_{11}\hat{x}_1 + A_{12}\hat{x}_2 + B_1u + v_0 + v_1 \\ \dot{\hat{x}}_2 &= A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + B_2u + L_2v_1 \\ v_0 &= L_1\tilde{x}_1 \\ v_1 &= \mathcal{ST}\{\sigma\} \\ \sigma &= \tilde{x}_1 + z, \end{aligned} \tag{6}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the estimates of x_1 and x_2, respectively; $\tilde{x}_1 = x_1 - \hat{x}_1$ is the estimation error variable; $v_0 \in \mathbb{R}^k$ and $v_1 \in \mathbb{R}^k$ are the observer input injections; $\sigma \in \mathbb{R}^k$ is the sliding variable and $z \in \mathbb{R}^k$ is an integral variable to be defined thereafter. Finally, $L_1 \in \mathbb{R}^{k\times k}$ and $L_2 \in \mathbb{R}^{(n-k)\times k}$ are the observer gains.

3.2 Convergence Analysis

Define the estimation error variable $\tilde{x}_2 = x_2 - \hat{x}_2$. From (5) and (6), it follows

$$\begin{aligned}\dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 - v_0 - v_1 \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 - L_2 v_1.\end{aligned} \tag{7}$$

First, note that the σ-dynamics are given by:

$$\begin{aligned}\dot{\sigma} &= \dot{\tilde{x}}_1 + \dot{z} \\ &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 - v_0 - v_1 + \dot{z}.\end{aligned} \tag{8}$$

Define now $\dot{z} = -A_{11}\tilde{x}_1 + v_0$, then

$$\dot{\sigma} = A_{12}\tilde{x}_2 - v_1. \tag{9}$$

The term $A_{12}\tilde{x}_2$ is assumed to be an unknown disturbance but with bounded time derivative, with $\left|\left|\frac{d}{dt}\left[A_{12}\tilde{x}_2\right]\right|\right| < \delta$ and $\delta > 0$ is a known positive constant. Then, since $v_1 = \mathcal{ST}\{\sigma\}$, it follows that $(\sigma(t), q) = (0, 0)\ \forall t > t_q$, with $q = w - A_{12}\tilde{x}_2$.

From the above analysis and (9), it follows that the equivalent control of v_1 (Utkin, 1992) is

$$\{v_1\}_{eq} = A_{12}\tilde{x}_2,$$

which implies that the motion of the system (7) constrained to the sliding manifold $(\sigma, q) = (0, 0)$ is given by:

$$\begin{aligned}\dot{\tilde{x}}_1 &= \left(A_{11} - L_1 I_k\right)\tilde{x}_1 \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + \left(A_{22} - L_2 A_{12}\right)\tilde{x}_2.\end{aligned} \tag{10}$$

where $I_k \in \mathbb{R}^{k\times k}$ is the k-order identity matrix. Hence, the system (10) associated eigenvalues are given by

$$\det\begin{bmatrix}\lambda I_k - (A_{11} - L_1 I_k) & 0 \\ -A_{21} & \lambda I_{n-k} - (A_{22} - L_2 A_{12})\end{bmatrix} = \det\left[\lambda I_k - (A_{11} - L_1 I_k)\right]\det\left[\lambda I_{n-k} - (A_{22} - L_2 A_{12})\right].$$

Since the pair (A_{11}, I_k) is always observable, it is possible to choose the gain L_1 so the matrix $A_{11} - L_1 I_k$ be Hurwitz. On the other hand, since the pair (A, C) was assumed to be observable, it can be shown that the pair (A_{22}, A_{12}) is also observable (Drakunov and Utkin, 1995; Shtessel et al., 2013). Then, the gain L_2 can be chosen so the matrix $A_{22} - L_2 A_{12}$ be Hurwitz. Hence, $\tilde{x}_1, \tilde{x}_2 \to 0$ as $t \to \infty$, and the convergence analysis is completed.

Remark 3.1. It is important to note that, with the proposed observer scheme (6), the dynamic behavior of the estimation blocks $\tilde{x}_1$ and $\tilde{x}_2$ can be tuned independently (see (10)).

4. DESIGN EXAMPLE

To verify the proposed observer performance, it will be applied to the following DC motor model (Utkin and Shi, 1996):

$$\begin{aligned}\dot{i} &= \frac{-R}{L}i - \frac{\lambda}{L}\omega + \frac{1}{L}V \\ \dot{\omega} &= \frac{K}{J}i - \frac{b}{J}\omega \\ y &= i\end{aligned} \tag{11}$$

where i is armature current, V is terminal voltage, ω is shaft speed, R is armature resistance, L is armature inductance, J is moment of inertia of the rotor, b is motor viscous friction constant and λ is back-EMF constant. Finally the measurable output of system is the armature current i.

Note that the DC motor model (11) has the form (5), with $i = x_1 = y$ and $\omega = x_2$; $A_{11} = -\frac{R}{L}$, $A_{12} = -\frac{\lambda}{L}$, $A_{21} = \frac{K}{J}$ and $A_{22} = -\frac{b}{J}$; $B_1 = \frac{1}{L}$ and $B_2 = 0$. Then, the integral SMO is given by (6), with $\dot{z} = -A_{11}\tilde{x}_1 + v_0$.

The simulation results for this design example are shown in the next section.

5. SIMULATION RESULTS

All simulations presented here were conducted using the Euler integration method with a fundamental step size of 1×10^{-3} [s]. The DC Motor parameters are shown in Table 1 (Utkin et al., 1999).

Table 1. Nominal Parameters of the DC motor model (11).

Parameter	Values	Unit
L	0.001	V
R	0.5	Ω
λ	0.001	$\mathrm{V \cdot s \cdot rad^{-1}}$
b	0.001	$\mathrm{N \cdot m \cdot s \cdot rad^{-1}}$
k	0.008	$\mathrm{N \cdot m \cdot A^{-1}}$
J	0.001	$\mathrm{kg \cdot m^2}$

The initial conditions for the system (11) were selected as: $i(0) = 31.5\,\mathrm{A}$ and $\omega(0) = 250\,\mathrm{rad/s}$; furthermore, for the designed observer in the form (6), the initial conditions were chosen as: $\hat{i}(0) = 25.2\,\mathrm{A}$, $\hat{\omega}(0) = 200\,\mathrm{rad/s}$, $z(0) = 0$ and $w(0) = 0$. In addition, applying super twisting algorithm (2), the parameter values for the observer were adjusted as: $L_1 = 2 \times 10^{-4}$, $L_2 = -0.01$, $\alpha_1 = 4.7434$ and $\alpha_2 = 11$.

This section is divided into two parts. In the first part, there is assumed that the current measurements are noiseless; in the second part instead, there is included a normally distributed random signal as measurement noise in the current. The applied voltage V is a DC source, with a magnitude of 16 V, which is suddenly reduced to 15 V at $t = 25$ [s].

5.1 Noiseless Measurements

In this subsection, there is assumed no noise in the current measurements. The following results were obtained:

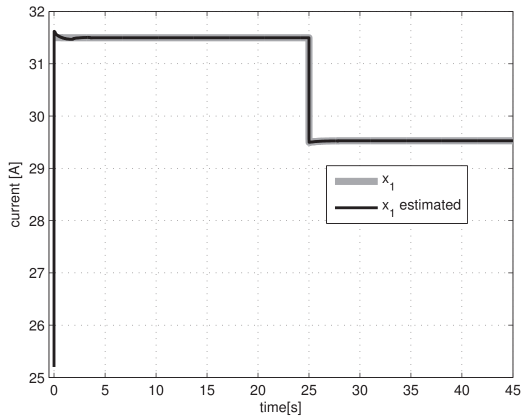

Figure 1. Armature current (i) (actual and estimated).

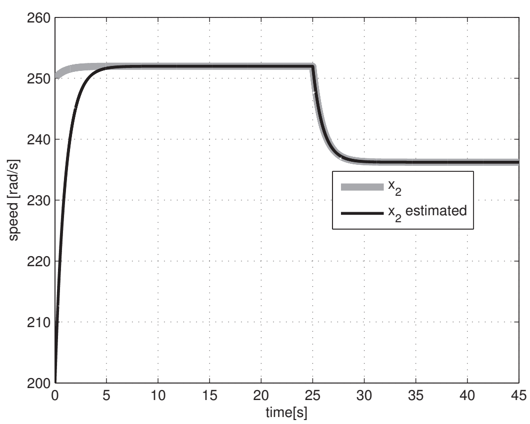

Figure 2. Shaft speed (ω) (actual and estimated).

Fig. 1 and Fig. 2 show the comparison between the actual and estimated variables corresponding to the armature current i and shaft speed ω respectively, for noiseless measurements.

5.2 Noisy Measurements

In this subsection, it is assumed that the current measurements were corrupted by a normally distributed random signal with zero mean and a variance of 10. This assumed variance corresponds to a current sensor with an accuracy of ± 9.5 A. This large variance was assumed to see significant variations in the simulation due to the noise and verify the filtering capabilities of the proposed observer. The following results were obtained:

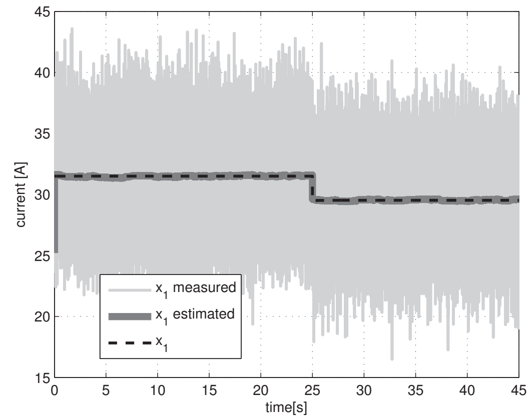

Figure 3. Armature current (i) (measured, estimated and actual).

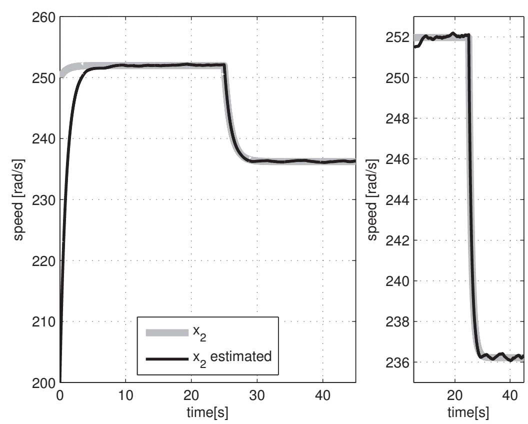

Figure 4. Shaft speed (ω) (actual and estimated).

Fig. 3 and Fig. 4 show the comparison between the actual and estimated variables corresponding to the armature current i and shaft speed ω respectively, for noisy measurements.

Based on the presented figures, it can be observed a good performance of the proposed observer. Under noisy conditions the armature current estimation $\hat{i}$ is much closer to its actual value than its measurement (Fig 3). In addition, a correct estimation of ω using the integral sliding mode observer is achieved (Figs. 2, 4) making the proposed observer suitable for observer-based control applications.

6. CONCLUSION

In this paper an integral sliding mode observer for linear time-invariant systems is proposed. The convergence of the estimation errors to zero for the proposed observer was proved. A step by step design of the proposed observer was provided along with a design example over a DC motor model. The simulation results of the example shown the filtering capabilities of the proposed observer.

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