Predefined-Time Backstepping Control for Tracking a Class of Mechanical Systems ⋆

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Abstract: The predefined-time exact tracking of unperturbed fully actuated mechanical systems is considered in this paper. A continuous second-order predefined-time stabilizing backstepping controller, designed using first-order predefined-time stabilizing functions, is developed to solve this problem. As an example, the proposed solution is applied over a two-link planar manipulator and numerical simulations are conducted to show performance of the proposed control scheme.

Keywords: Backstepping, Lyapunov, Mechanical systems, Predefined-time stability, Second-order systems, Tracking.

1. INTRODUCTION

The various developments concerning the concept of *finite-time stability* permit to solve different applications which are characterized for requiring hard time response constraints. Some important works of this topic and its application to control systems have been carried out in Roxin (1966); Haimo (1986); Utkin (1992); Bhat and Bernstein (2000); Moulay and Perruquetti (2005, 2006).

However, generally this finite time is an unbounded function of the initial conditions of the system. A desired feature is to eliminate this boundlessness, for example, in estimation or optimization problems. This gives rise to a stronger form of stability called *fixed-time stability*, where the convergence time, as a function of the initial conditions, is bounded. The notion of fixed-time stability have been investigated in Andrieu et al. (2008); Cruz-Zavala et al. (2010); Polyakov (2012); Fraguela et al. (2012); Polyakov and Fridman (2014).

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time. To overcome the above, another class of dynamical systems which exhibit the property of *predefined-time stability*, have been studied (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015). For this systems, an upper bound (sometimes the least upper bound) of the fixed stabilization time appears explicitly in their tuning gains.

In this sense, similarly to Jiménez-Rodríguez et al. (2016); Sánchez-Torres et al. (2016), this paper is devoted to the design of a *second-order predefined-time controller*, using first-order predefined-time stabilizing functions (Sánchez-Torres et al., 2015) and the backstepping design technique (Kokotovic, 1992; Krstić et al., 1995). Furthermore, this idea is used to solve the problem of predefined-time exact tracking in fully actuated mechanical systems, assuming the availability of the state and the desired trajectory (as well as its two first derivatives) measurements.

In the following, Section 2 presents the mathematical preliminaries needed to introduce the proposed results. Section 3 states the problem which will be solved in this paper. Section 4 exposes the main result of this paper, which is the second-order predefined-time backstepping controller for tracking of fully actuated mechanical systems. Section 5 describes the model of a planar two-link manipulator, where the proposed controller is applied. The simulation results of the example are shown in Section 6. Finally, Section 7 presents the conclusions of this paper.

2. MATHEMATICAL PRELIMINARIES

Consider the system

$$\dot{x} = f(x; \rho) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system and $f : \mathbb{R}^n \to \mathbb{R}^n$. The initial conditions of this system are $x(0) = x_0$.

Definition 1. (Bhat and Bernstein, 2000; Polyakov, 2012) The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the *settling-time function*.

Remark 2. The settling time function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ for systems with a finite-time stable equilibrium point

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is usually an unbounded function of the system initial condition.

Definition 3. (Polyakov, 2012; Polyakov and Fridman, 2014) The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \le T_{\max}$.

Definition 4. (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015) For the case of fixed time stability when the system (1) parameters ρ can be expressed in terms of a bound of the settling-time function T_{max}, it is said that the origin of the system (1) is *predefined-time stable.*

The following lemma extends the Lyapunov methods given in Sánchez-Torres et al. (2014); Sánchez-Torres et al. (2015). This result will be useful in order to apply the approaches presented in the mentioned references to second order systems.

Lemma 5. Assume there exist a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, and real numbers $\alpha > 0$, $\beta > 0$, and $0 < q \le 1$ such that:

$$V(0) = 0$$
$$V(x) > 0, \qquad \forall x \neq 0,$$

and the derivative of V along the trajectories of the system (1) satisfies

$$\dot{V} \le -\frac{\alpha}{q} \exp(\beta V^q) V^{1-q}. \tag{2}$$

Then, the origin is globally predefined-time stable for (1) and $T(x_0) \le \frac{1}{\alpha\beta}$.

Proof. The solution of the differential inequality (2) is

$$V(t) \le \left[\frac{1}{\beta} \ln \left(\frac{1}{\alpha\beta t + \exp(-\beta V_0^q)} \right) \right]^{\frac{1}{q}},$$

where $V_0 = V(x_0)$.

Note that $V(t) = 0$ if $\alpha t + \exp(-\beta V_0^q) = 1$. Hence, the settling-time function is such that

$$T(x_0) \le \frac{1 - \exp(-\beta V_0^q)}{\alpha\beta}.$$

Then, $T(x_0) \le \frac{1}{\alpha\beta}$, since $0 < \exp(-\beta V_0^q) \le 1$.

■

Remark 6. Lemma 5 characterizes predefined-time stability in a very practical way since the condition (2) directly involves a bound on the convergence time.

Definition 7. Let $h \ge 0$. For $\boldsymbol{x} \in \mathbb{R}^n$, define the function

$$||\lfloor x \rceil||^h = \frac{x}{||x||^{1-h}}, \tag{3}$$

with $||x||$ the norm of x. Since $\lim_{x \to 0} ||\lfloor x \rceil||^h = 0$ for $h > 0$, it is defined $||\lfloor 0 \rceil||^h = 0$. Therefore, the function $||\lfloor x \rceil||^h$ is continuous for $h > 0$ and discontinuous in $x = 0$ for $h = 0$.

Theorem 8. The function $||\lfloor x \rceil||^h$ fulfills:

(i) $||\lfloor -x \rceil||^h = -||\lfloor x \rceil||^h$
(ii) $||\lfloor x \rceil||^0 = \frac{x}{||x||}$, a unit vector.
(iii) $||\lfloor x \rceil||^1 = ||\lfloor x \rceil|| = x$,
(iv) $\frac{d||\lfloor x \rceil||^h}{dx} = \left[I_n + (h-1)\frac{xx^T}{||x||^2} \right] ||x||^{h-1}$ and $\frac{d||x||^h}{dx} = h\left|\left|\lfloor x^T \rceil\right|\right|^{h-1}$, where I_n is the $n \times n$ identity matrix.
(v) For $h_1, h_2 \in \mathbb{R}$, it follows:
· $||x||^{h_1} ||x||^{h_2} = ||x||^{h_1+h_2}$
· $\left|\left|\lfloor x^T \rceil\right|\right|^{h_1} ||\lfloor x \rceil||^{h_2} = ||x||^{h_1+h_2}$
(vi) For $h_1, h_2 > 0$, then $\left|\left|\left\lfloor ||\lfloor x \rceil||^{h_1} \right\rceil\right|\right|^{h_2} = ||\lfloor x \rceil||^{h_1 h_2}$.

Definition 9. (Sánchez-Torres et al., 2015) For $x \in \mathbb{R}^n$, the *predefined-time stabilizing function* is defined as

$$\Phi_{m,q}(x; T_c) = \frac{1}{T_c m q} \exp\left(||x||^{mq}\right) ||\lfloor x \rceil||^{1-mq} \tag{4}$$

where $m \ge 1$, $0 < q \le \frac{1}{m}$, and $T_c > 0$.

Remark 10. From Definition 7, the function defined in (4) is continuous for $0 < mq < 1$ and discontinuous in $x = 0$ for $mq = 1$.

Remark 11. Using the part *(iv)* Theorem 8, the derivative of the predefined-time stabilizing function (4) is given by

$$\frac{\partial \Phi_{m,q}(x; T_c)}{\partial x} = \frac{\exp\left(||x||^{mq}\right)}{T_c m q} \left[mq\frac{xx^T}{||x||^2} + \left(I_n - mq\frac{xx^T}{||x||^2} \right) \frac{1}{||x||^{mq}} \right], \tag{5}$$

for $x \neq 0$.

The following lemma gives meaning to the name predefined-time stabilizing function, introduced in Definition 9.

Lemma 12. (Sánchez-Torres et al., 2015) The origin of the system

$$\dot{x} = -\Phi_{m,q}(x; T_c) \tag{6}$$

with $m \ge 1$, $0 < q \le \frac{1}{m}$, and $T_c > 0$ is predefined-time stable with $T_f = T_c$. That is, $x(t) = 0$ for $t > T_c$ in spite of the x_0 value.

Proof. Consider the radially unbounded Lyapunov function candidate $V(x) = ||x||^m$. The time derivative of V along the trajectories of (6) is (see Theorem 8)

$$\begin{aligned} \dot{V} &= -m\left|\left|\lfloor x^T \rceil\right|\right|^{m-1} \Phi_{m,q}(x; T_c) \\ &= -\frac{m}{T_c m q} \exp\left(||x||^{mq}\right) ||x||^{m(1-q)} \\ &= -\frac{1}{T_c q} \exp\left(V^q\right) V^{1-q}. \end{aligned}$$

Hence, applying Lemma 5, the origin of the system (6) is predefined-time stable with $T_f = T_c$.

■

3. PROBLEM STATEMENT

A generic model of second-order, fully actuated mechanical systems of n degrees of freedom has the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + P(\dot{q}) + \gamma(q) = \tau, \tag{7}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position, velocity and acceleration vectors in joint space; $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis and centrifugal effects matrix, $P(\dot{q}) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction and $\gamma(q) \in \mathbb{R}^n$ is the gravity effects vector.

Defining the variables $x_1 = q$, $x_2 = \dot{q}$ and $u = \tau$, the mechanical model (7) can be rewritten in the following state-space form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) + B(x_1, x_2)u,\end{aligned} \tag{8}$$

where $f(x_1, x_2) = -M^{-1}(x_1)[C(x_1, x_2)x_2 + P(x_2) + \gamma(x_1)]$, $B(x_1, x_2) = M^{-1}(x_1)$ are continuous maps and the initial conditions are $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$.

Remark 13. The matrix function $M(x_1)$ is, in fact, invertible since $M(x_1) = M^T(x_1)$ is positive definite.

A common problem in mechanical systems control is to track a desired time-dependent trajectory described by the triplet $(q_d(t), \dot{q}_d(t), \ddot{q}_d(t))$ of desired position $q_d(t) = [q_{d_1}(t) \ \cdots \ q_{d_n}(t)]^T \in \mathbb{R}^n$, velocity $\dot{q}_d(t) = [\dot{q}_{d_1}(t) \ \cdots \ \dot{q}_{d_n}(t)]^T \in \mathbb{R}^n$ and acceleration $\ddot{q}_d(t) = [\ddot{q}_{d_1}(t) \ \cdots \ \ddot{q}_{d_n}(t)]^T \in \mathbb{R}^n$, which are all assumed to be known.

The task is to design a state-feedback, second-order, predefined-time controller to track the desired trajectory.

4. PREDEFINED-TIME BACKSTEPPING TRACKING CONTROLLER

To be consequent with the state-space notation used in (8), redefine the desired position as $x_{1,d} = q_d$.

Step 1: let the position error be $e_1 = x_1 - x_{1,d}$. Then, using (8), the dynamics of e_1 are

$$\dot{e}_1 = x_2 - \dot{x}_{1,d}. \tag{9}$$

Let $m_1 \geq 1$ and consider the Lyapunov function candidate

$$V_1(e_1) = ||e_1||^{m_1}. \tag{10}$$

Differentiating (10) with respect to time, along the trajectories of (9), it yields

$$\dot{V}_1 = m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} [x_2 - \dot{x}_{1,d}]. \tag{11}$$

Let

$$e_2 = x_2 - x_{2,d}, \tag{12}$$

with $x_{2,d} = -\Phi_{m_1,q_1}(e_1; T_{c_1}) + \dot{x}_{1,d}$, where $0 < q_1 < \frac{1}{2m_1}$ and $T_{c_1} > 0$.

Replacing (12), (11) becomes

$$\begin{aligned}\dot{V}_1 &= -m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} \Phi_{m_1,q_1}(e_1; T_{c_1}) + m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} e_2 \\ &= -\frac{1}{T_c q_1} \exp\left(V_1^{q_1}\right) V_1^{1-q_1} + m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} e_2.\end{aligned} \tag{13}$$

Step 2: using (8), the dynamics of (12) are

$$\dot{e}_2 = f(x_1, x_2) + B(x_1, x_2)u + \frac{d\Phi_{m_1,q_1}(e_1; T_{c_1})}{dt} - \ddot{x}_{1,d}. \tag{14}$$

Let $1 \leq m_2 \leq 2$ and consider the Lyapunov function candidate

$$V(e_1, e_2) = ||e_1||^{m_1} + ||e_2||^{m_2} = V_1 + ||e_2||^{m_2}. \tag{15}$$

Differentiating (15) with respect to time, and substituting (13) and (9) it yields

$$\begin{aligned}\dot{V} = &-m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} \Phi_{m_1,q_1}(e_1; T_{c_1}) + m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} e_2 + \\ &m_2 \left|\left\lfloor e_2^T \right\rceil\right|^{m_2 - 1} \Bigg[f(x_1, x_2) + B(x_1, x_2)u + \\ &\frac{d\Phi_{m_1,q_1}(e_1; T_{c_1})}{dt} - \ddot{x}_{1,d} \Bigg].\end{aligned} \tag{16}$$

Thus, the control variable u is designed as

$$\begin{aligned}u = B^{-1}(x_1, x_2) \Bigg[&-f(x_1, x_2) - \frac{d\Phi_{m_1,q_1}(e_1; T_{c_1})}{dt} + \ddot{x}_{1,d} - \\ &\frac{m_1}{m_2} \left|\left\lfloor e_2 \right\rceil\right|^{1-m_2} \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} e_2 - \Phi_{m_2,q_2}(e_2; T_{c_2}) \Bigg],\end{aligned} \tag{17}$$

where $0 < q_2 < \frac{1}{m_2}$ and $T_{c_2} > 0$.

Finally, substituting (17) in (16) it yields

$$\begin{aligned}\dot{V} = &-m_1 \left|\left\lfloor e_1^T \right\rceil\right|^{m_1 - 1} \Phi_{m_1,q_1}(e_1; T_{c_1}) - \\ &m_2 \left|\left\lfloor e_2^T \right\rceil\right|^{m_2 - 1} \Phi_{m_2,q_2}(e_2; T_{c_2}) < 0.\end{aligned} \tag{18}$$

From (18), the error system (9) and (14) closed-loop with the control signal (17) is asymptotically stable. Moreover, the predefined-time stability is stated in Theorem 14 and proved in Appendix B.

Theorem 14. The origin of the error system (9) and (14), closed-loop with (17) is globally predefined-time stable with $T_f \leq 2rT_c$, where $T_c = \max\{T_{c_1}, T_{c_2}\}$, $r = \frac{q_M}{q_m}$, $q_m = \min\{q_1, q_2\}$ and $q_M = \max\{q_1, q_2\}$.

Remark 15. Note that selecting $q_1 = q_2$, the number r is minimized and becomes $r = 1$. In the same manner, the number T_c is minimized selecting $T_{c_1} = T_{c_2} = T_c$.

5. EXAMPLE: TRAJECTORY TRACKING FOR A TWO-LINK MANIPULATOR

Consider a planar, two-link manipulator with revolute joints as the one exposed in Example 12.1 of Utkin et al. (2009). The manipulator link lengths are L_1 and L_2, the link masses (concentrated in the end of each link) are M_1 and M_2. The manipulator is operated in the plane, such that the gravity acts along the z−axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass M_2 is concentrated) position (x_w, y_w) is given by

$$\begin{aligned}x_w &= L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) \\ y_w &= L_1 \sin(q_1) + L_2 \sin(q_1 + q_2),\end{aligned} \tag{19}$$

where q_1 and q_2 are the joint positions (angular positions).

Applying the Euler-Lagrange equations, a model according to (7) is obtained, with

$$\begin{aligned}m_{11} &= L_1^2(M_1 + M_2) + 2(L_2^2 M_2 + L_1 L_1 M_2 \cos q_2) - L_2^2 M_2 \\ m_{12} &= m_{21} = L_2^2 M_2 + L_1 L_1 M_2 \cos q_2 \\ m_{22} &= L_2^2 M_2 \\ h &= L_1 L_2 M_2 \sin q_2 \\ c_{11} &= -h\dot{q}_2 \\ c_{12} &= -h(\dot{q}_1 + \dot{q}_2) \\ c_{21} &= h\dot{q}_1 \\ c_{22} &= 0,\end{aligned}$$

$$M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad C(q, \dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$P(\dot{q}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \gamma(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius r_d and center in the origin. To solve this problem the controller exposed in Section 4 is applied.

6. SIMULATION RESULTS

The simulation results of the example in Section 5 are presented in this section. The two-link manipulator parameters used are shown in Table 1.

Table 1. Parameters of the two-link manipulator model.

Parameter	Values	Unit
M_1	0.2	kg
M_2	0.2	kg
L_1	0.2	m
L_2	0.2	m

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The initial conditions for the two-link manipulator were selected as: $x_1(0) = \begin{bmatrix} -\frac{3\pi}{4} & -\frac{\pi}{4} \end{bmatrix}^T$ and $x_2(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. In addition, the controller gains were adjusted to: $T_{c_1} = T_{c_2} = 1.5$, $m_1 = m_2 = 2$, and $q_1 = q_2 = \frac{1}{6}$.

The desired circular trajectory in the joint coordinates is described by the equations

$$q_d(t) = x_{1,d}(t) = \begin{bmatrix} q_{d_1}(t) \\ q_{d_2}(t) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2}t - \pi \\ \\ -\frac{\pi}{2} \end{bmatrix}, \qquad (20)$$

and it corresponds to a circumference of radius 0.2828 m.

The following figures show the behavior of the proposed controller.

Note that $V(t) = 0$ for $t \geq 1.3$ s $< 2T_c = 3$ s (Fig. 1), which implies that the error variables are exactly zero at the same time (Fig. 2). Fig. 3 shows the control signal (torque) versus time. Finally, from Fig. 4, it can be seen the reference tracking in rectangular coordinates.

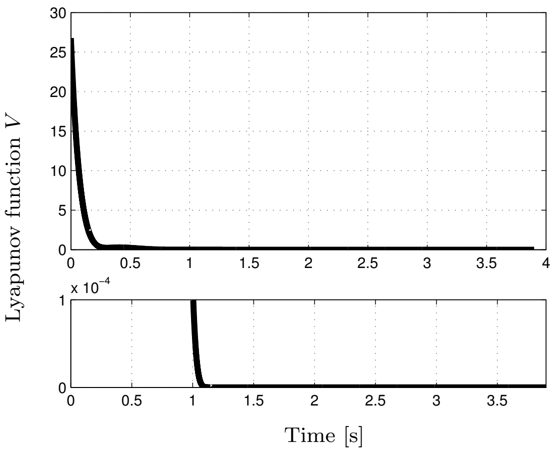

Fig. 1. Lyapunov function V. Note that $V(t) = 0$ for $t > 2T_c = 3$ s.

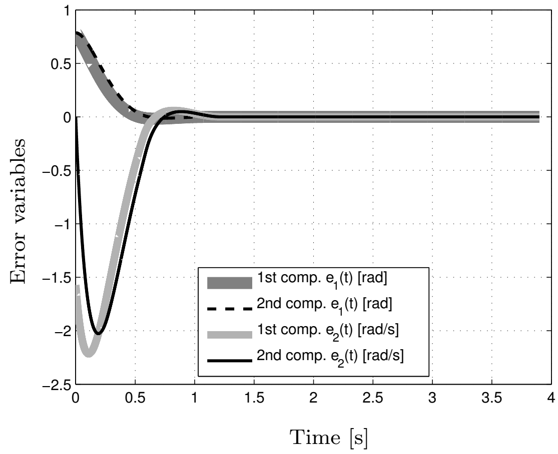

Fig. 2. Error variables. First component of e_1 (dark gray and thick), second component of e_1 (black and dashed), first component of e_2 (light gray and solid) and second component of e_2 (black and solid).

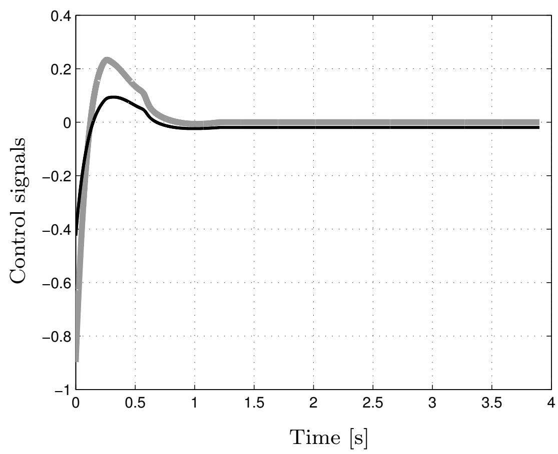

Fig. 3. Control signal. First component (gray and solid) and second component (black and solid).

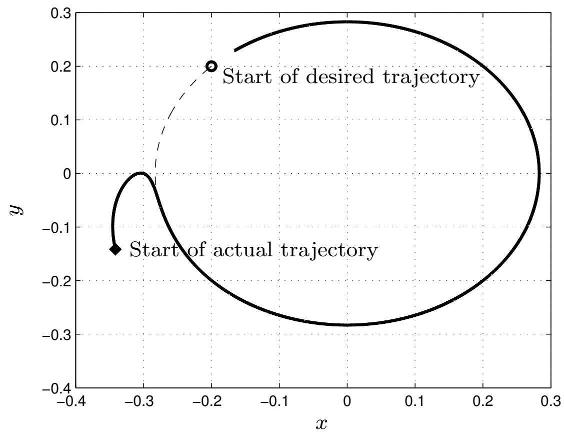

Fig. 4. Actual trajectory (x_w, y_w) (black and solid) and desired trajectory $(x_{w,d}, y_{w,d})$ (black and dashed).

7. CONCLUSION

In this paper the problem of predefined-time exact tracking in fully actuated mechanical systems was solved by means of a continuous second-order predefined-time backstepping controller. This controller was constructed as an application of continuous first-order predefined-time stabilizing functions. To show the feasibility of the proposed controller, it was implemented over a two-link planar manipulator. The numerical simulations showed a good performance.

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Appendix A. SOME IMPORTANT INEQUALITIES

In this appendix, some important inequalities are reviewed. These results were taken from (Hardy et al., 1934) and (Mitrinovic, 1970).

A.1 Chebyshev's inequality

Theorem 16. Let $n \in \mathbb{N}$. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two similarly ordered real sequences, i.e.,

$$a_1 \leq \cdots \leq a_n \text{ and } b_1 \leq \cdots \leq b_n$$

or

$$a_1 \geq \cdots \geq a_n \text{ and } b_1 \geq \cdots \geq b_n.$$

Then, the following inequality holds

$$\sum_{i=1}^{n} a_i b_i \geq \frac{1}{n} \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Corollary 17. Let $x_1, x_2 \in \mathbb{R}$ be two real numbers and $f, g : \mathbb{R} \to \mathbb{R}$ be two similarly ordered functions, i.e., if $x_1 \leq x_2$, then

$$f(x_1) \leq f(x_2) \text{ and } g(x_1) \leq g(x_2),$$

or

$$f(x_1) \geq f(x_2) \text{ and } g(x_1) \geq g(x_2).$$

Then,

$$f(x_1)g(x_1) + f(x_2)g(x_2) \geq \frac{1}{2}\left(f(x_1) + f(x_2)\right)\left(g(x_1) + g(x_2)\right).$$

A.2 Inequality of arithmetic and geometric means

Theorem 18. Let $n \in \mathbb{N}$. If $a = (a_1, \ldots, a_n)$ is a sequence of positive numbers, then

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

Corollary 19. Let $x_1, x_2 \in \mathbb{R}_+$ be two positive real numbers, then

$$x_1 + x_2 \geq 2\left(x_1 x_2\right)^{\frac{1}{2}}.$$

Appendix B. PROOF OF THEOREM 14

Some lemmas are to be stated and proved before proving Theorem 14.

Lemma 20. Let $0 < q \leq 1$ and consider the function $f_q : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ defined by

$$f_q(x) = \exp\left(x^q\right) x^{1-q}.$$

Then, the function f_q is increasing with respect to q, i.e., if $0 < q_1 \leq q_2 \leq 1$, then for $x \in \mathbb{R}_+ \cup \{0\}$

$$f_{q_1}(x) \leq f_{q_2}(x).$$

Proof. Note that for any $0 < q \leq 1$, $f_q(0) = 0$. Then, for $0 < q_1 \leq q_2 \leq 1$

$$f_{q_1}(0) = 0 \leq 0 = f_{q_2}(0).$$

Furthermore,

$$\frac{\partial f_q(x)}{\partial q} = \exp\left(x^q\right)\left[x - x^{1-q}\right]\ln(x) \begin{cases} > 0, & 0 < x < 1 \\ = 0, & x = 1 \\ > 0, & x > 1. \end{cases}$$

In any case $\frac{\partial f_q(x)}{\partial q} \geq 0$, i.e., the function f_q is increasing with respect to q.

■

Lemma 21. Let $0 < q \leq 1$ and $x_1, x_2 \in \mathbb{R}_+ \cup \{0\}$. Then the following inequality holds

$$\exp\left(x_1^q\right) x_1^{1-q} + \exp\left(x_2^q\right) x_2^{1-q} \geq \frac{1}{2}\left[\exp\left(x_1^q\right) + \exp\left(x_2^q\right)\right]\left[x_1^{1-q} + x_2^{1-q}\right].$$

Proof. Define the functions $f(x) = \exp\left(x^q\right)$ and $g(x) = x^{1-q}$. Note that

$$\frac{df(x)}{dx} = q\exp\left(x^q\right) x^{q-1} > 0 \text{ and } \frac{dg(x)}{dx} = (1-q)x^{-q} > 0.$$

Then, since both functions f, g are increasing, the result is a direct consequence of Corollary 17.

■

Lemma 22. Let $0 \leq q \leq 1$ and $x_1, x_2 \in \mathbb{R}_+ \cup \{0\}$. Then,

$$x_1^q + x_2^q \geq (x_1 + x_2)^q.$$

Proof. Consider the function $\epsilon : \{\mathbb{R}_+ \cup \{0\}\} \times \{\mathbb{R}_+ \cup \{0\}\} \to \mathbb{R}$, defined by $\epsilon(x, y) = x^q + y^q - (x+y)^q$. It is to be proved that $\epsilon(x_1, x_2) \geq 0$.

First of all, note that $\epsilon(x, 0) = \epsilon(0, y) = 0$ for all $x, y \in \mathbb{R}_+ \cup \{0\}$. Furthermore, since $-1 \leq q - 1 \leq 0$, the partial derivatives

$$\frac{\partial \epsilon(x, y)}{\partial x} = q\left[x^{q-1} - (x+y)^{q-1}\right] \geq 0,$$

and

$$\frac{\partial \epsilon(x, y)}{\partial y} = q\left[y^{q-1} - (x+y)^{q-1}\right] \geq 0,$$

for $x, y \in \mathbb{R}_+$.

Hence, $\epsilon(x_1, x_2) \geq 0$ and the proof is concluded.

■

With the above results,

Proof. (Of Theorem 14) From (18)

$$\dot{V} = -\frac{1}{T_{c_1} q_1}\exp\left(||e_1||^{m_1 q_1}\right)||e_1||^{m_1(1-q_1)} - \frac{1}{T_{c_2} q_2}\exp\left(||e_2||^{m_2 q_2}\right)||e_2||^{m_2(1-q_2)}.$$

Defining $T_c = \max\{T_{c_1}, T_{c_2}\}$ and $q_M = \max\{q_1, q_2\}$, the following holds

$$\dot{V} \leq -\frac{1}{T_c q_M}\left[\exp\left(||e_1||^{m_1 q_1}\right)||e_1||^{m_1(1-q_1)} + \exp\left(||e_2||^{m_2 q_2}\right)||e_2||^{m_2(1-q_2)}\right].$$

Now, defining $q_m = \min\{q_1, q_2\}$ and applying Lemma 20 to the right side of the above inequality, it yields

$$\dot{V} \leq -\frac{1}{T_c q_M}\left[\exp\left(||e_1||^{m_1 q_m}\right)||e_1||^{m_1(1-q_m)} + \exp\left(||e_2||^{m_2 q_m}\right)||e_2||^{m_2(1-q_m)}\right].$$

Using Lemma 21, the following inequality is obtained

$$\dot{V} \leq -\frac{1}{2T_c q_M}\left[\exp\left(||e_1||^{m_1 q_m}\right) + \exp\left(||e_2||^{m_2 q_m}\right)\right] \times \left[||e_1||^{m_1(1-q_m)} + ||e_2||^{m_2(1-q_m)}\right].$$

Since $\exp(x) > 0$ for all $x \in \mathbb{R}$, a direct application of Corollary 19 yields

$$\dot{V} \leq -\frac{1}{T_c q_M}\exp\left[\frac{1}{2}\left(||e_1||^{m_1 q_m} + ||e_2||^{m_2 q_m}\right)\right] \times \left(||e_1||^{m_1(1-q_m)} + ||e_2||^{m_2(1-q_m)}\right).$$

Finally, defining $r = \frac{q_M}{q_m}$ and, since $0 < q_m \leq q_1 < \frac{1}{2} < 1$, using Lemma 22

$$\dot{V} \leq -\frac{1}{rT_c q_m}\exp\left[\frac{1}{2}\left(||e_1||^{m_1} + ||e_2||^{m_2}\right)^{q_m}\right] \times \left(||e_1||^{m_1} + ||e_2||^{m_2}\right)^{(1-q_m)}$$
$$= -\frac{1}{rT_c q_m}\exp\left(\frac{1}{2}V^{q_m}\right)V^{(1-q_m)}.$$

Hence, Theorem 14 follows from Lemma 5.

■