A Second Order Predefined-Time Control Algorithm

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***Abstract*—The predefined-time stabilization of second-order systems, i.e. the fixed-time stabilization with settling time as a function of the controller parameters, is revisited in this paper. The proposed controller is a time-based switched controller which first drive the system trajectories to a linear manifold in predefined time and then uses a nested second-order controller. The application of the results is demonstrated for the trajectory tracking control in fully actuated mechanical systems. An illustrative example of the control of a two-link planar manipulator with predefined-time convergence shows the effectiveness of the proposed algorithm.**

I. Introduction

Sliding mode techniques are based on the idea of driving the trajectory of a treated dynamical system to a specified manifold that is to be reached after a limited time period [1]. Thus, controllers and observers based on those methods are highly related to the concept of finite-time stability and can provide solutions to applications which require hard time response constraints. Significant works involving the definition and application of finite-time stability have been carried out in [2]–[7].

In spite of that, this finite stabilization time is often an unbounded function of the initial conditions of the system. To deal with this drawback a stronger form of stability, called *fixed-time stability*, was introduced [8]–[14], making this function of the initial conditions globally bounded to ensure the settling time is less than a certain quantity for any initial condition.

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time. To overcome the above, another class of dynamical systems which exhibit the property of *predefined-time stability*, have been studied [15]–[17]. For this systems, an upper bound (sometimes the least upper bound) of the fixed stabilization time appears explicitly in their tuning gains.

In this sense, the results [15]–[17] present first order predefined-time stable dynamical systems. Furthermore, the works [18], [19] attempt to extend the mentioned results to second-order systems as a nested application of first order predefined-time stabilizing functions. However, since the predefined-time stabilizing function is non-smooth, these approaches yield a singular controller which may produce theoretically infinite signals.

In this paper, the region where the controller developed in [18] does not undergo singularities is estimated. Moreover, a time-based switched controller is proposed to first drive the system trajectories to the estimated region in predefined time and then use the controller in [18]. Furthermore, this idea is used to solve the problem of predefined-time exact tracking in fully actuated mechanical systems, assuming the availability of the state and the desired trajectory measurements. As a case study, the controller is applied for the predefined-time exact trajectory tracking in a planar two-link manipulator.

II. Preliminaries

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{\rho}), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, $\boldsymbol{\rho} \in \mathbb{R}^b$ represents the parameters of the system and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. The initial conditions of this system are $\boldsymbol{x}_0 = \boldsymbol{x}(0)$.

Definition 1 (*Global finite-time stability* [5], [10])**.** The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(\boldsymbol{x}_0) : \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{0}$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the *settling-time function*.

Definition 2 (*Fixed-time stability* [10], [11])**.** The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\text{max}} > 0 : \forall \boldsymbol{x}_0 \in \mathbb{R}^n : T(\boldsymbol{x}_0) \leq T_{\text{max}}$.

Remark 1. Note that there are several possible choices for T_{max}; for example, if $T(\boldsymbol{x}_0) \leq T_{\text{m}}$ for a positive number T_m, also $T(\boldsymbol{x}_0) \leq \lambda T_{\text{m}}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition 3 (*Settling-time set and its minimum bound* [15], [16])**.** Let the origin be fixed-time-stable for the system (1). The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \left\{ T_{\text{max}} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\text{m}},\ \forall\, \boldsymbol{x}_0 \in \mathbb{R}^n \right\}. \tag{2}$$

In addition, the least upper bound of the settling-time function, denoted by T_f, is defined as

$$T_f = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0). \tag{3}$$

Remark 2. For several applications it could be desirable for system (1) to stabilize within a time $T_c \in \mathcal{T}$ which can be defined in advance as function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. The cases where this property is present motivate the definition of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, i.e., T_c is the true fixed-time in which the system stabilizes. A weak notion of predefined-time stability is presented when $T_c \geq T_f$, that is, if well it is possible to define an upper bound of the settling-time function in terms of the system parameters, this overestimates the true fixed-time in which the system stabilizes.

Definition 4 (*Predefined-time stability* [17])**.** For the system parameters $\boldsymbol{\rho}$ and a constant $T_c(\boldsymbol{\rho}) > 0$, the origin of (1) is said to be

(i) *Globally weakly predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

In this case, T_c is called a *weak predefined time*.

(ii) *Globally strongly predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c.$$

In this case, T_c is called the *strong predefined time*.

Definition 5 (*Predefined-time stabilizing function* [17])**.** For $\boldsymbol{x} \in \mathbb{R}^n$, the *predefined-time stabilizing function* is defined as

$$\boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) = \frac{1}{mqT_c} \exp\left(||\boldsymbol{x}||^{mq}\right) \frac{\boldsymbol{x}}{||\boldsymbol{x}||^{mq}}, \tag{4}$$

where $T_c > 0$, $m \geq 1$ and $0 < q \leq \frac{1}{m}$.

Proposition 1 (*Predefined-time stabilizing function derivative* [18])**.** *The derivative of the predefined-time stabilizing function* (4) *is given by*

$$\frac{\partial \boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c)}{\partial \boldsymbol{x}} = \frac{\exp\left(||\boldsymbol{x}||^{mq}\right)}{mqT_c} \left[mq \frac{\boldsymbol{x}\boldsymbol{x}^T}{||\boldsymbol{x}||^2} + \left(\boldsymbol{I}_n - mq \frac{\boldsymbol{x}\boldsymbol{x}^T}{||\boldsymbol{x}||^2} \right) \frac{1}{||\boldsymbol{x}||^{mq}} \right], \tag{5}$$

for all $\boldsymbol{x} \neq \boldsymbol{0}$, *where* $\boldsymbol{I}_n \in \mathbb{R}^{n \times n}$ *stands for the* $n-$*th order identity matrix.*

Note that (5) *is defined everywhere, except in* $\boldsymbol{x} = 0$.

The following two lemmas present dynamical systems with the predefined-time stability property. The predefined-time stabilizing function (4) plays a main role, which justifies its name.

Lemma 1 (*A strongly predefined-time stable dynamical system* [17])**.** *The origin of the system*

$$\dot{\boldsymbol{x}} = -\boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) \tag{6}$$

with $T_c > 0$, $m \geq 1$ *and* $0 < q \leq \frac{1}{m}$ *is globally strongly predefined-time stable with strong predefined time* T_c.

Lemma 2 (*A weakly predefined-time stable dynamical system* [17])**.** *Let the function* $\boldsymbol{\Delta}(t, \boldsymbol{x})$ *be considered as a non-vanishing bounded disturbance such that* $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq \delta$, *with* $0 < \delta < \infty$ *a known constant. The origin of the system*

$$\dot{\boldsymbol{x}} = -k \frac{\boldsymbol{x}}{||\boldsymbol{x}||} - \boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) + \boldsymbol{\Delta}(t, \boldsymbol{x}) \tag{7}$$

with $k \geq \delta$, $T_c > 0$, $m \geq 1$ *and* $0 < q \leq \frac{1}{m}$ *is globally weakly predefined-time stable with weak predefined time* T_c.

III. Motivation

Consider the scalar double-integrator system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned} \tag{8}$$

where $x_1, x_2, u \in \mathbb{R}$.

The objective is to make the origin of the system (8), $(x_1, x_2) = (0, 0)$, globally predefined-time stable. With basis on the predefined-time stabilizing function (4), a good candidate of desired compensated dynamics is

$$\dot{x}_1 + \Phi_{m_1,q_1}(x_1; T_{c_1}) = 0,$$

with $m_1 \geq 1$, $0 < q_1 < \frac{1}{m_1}$ and $T_{c_1} > 0$. Note that once the above desired compensated dynamics are achieved, using Lemma 1, $x_1(t) = 0$ for $t \geq T_{c_1}$. Furthermore, since $x_2 = \dot{x}_1$, $x_2(t) = 0$ for $t \geq T_{c_1}$ also.

Thus, the problem has been reduced to achieve the above desired compensated dynamics in predefined time. With this aim, let's introduce a new variable σ as

$$\sigma = x_2 + \Phi_{m_1,q_1}(x_1; T_{c_1}). \tag{9}$$

From (8) and (9), the dynamics of the variable σ is

$$\begin{aligned} \dot{\sigma} &= u + \frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1} x_2 \\ &= u + \frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1} (\sigma - \Phi_{m_1,q_1}(x_1; T_{c_1})). \end{aligned}$$

The control signal u is to be designed to stabilize σ in predefined time. As a first attempt, one may propose the following controller

$$\begin{aligned} u &= -\frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1} x_2 - \Phi_{m_2,q_2}(\sigma; T_{c_2}), \\ &= \frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1} (\Phi_{m_1,q_1}(x_1; T_{c_1}) - \sigma) - \Phi_{m_2,q_2}(\sigma; T_{c_2}) \end{aligned} \tag{10}$$

with $m_2 \geq 1$, $0 < q_2 \leq \frac{1}{m_2}$ and T_{c_2}, which is the main idea of the approach presented in [18]. Here, some things should be noticed:

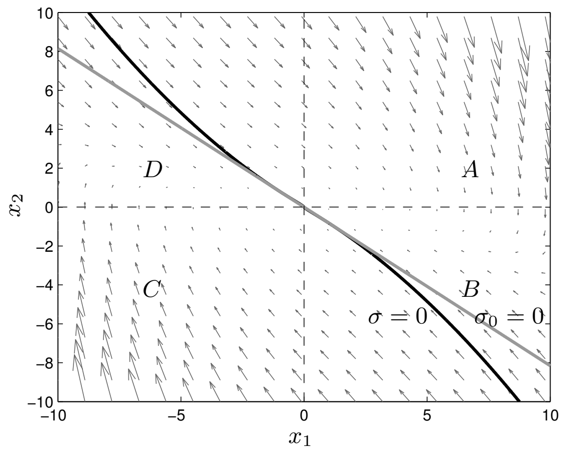

Fig. 1: Phase portrait of the closed-loop system (8)-(10) (gray arrows), manifold $\sigma = 0$ (black line) and manifold $\sigma_0 = 0$ (gray line).

(i) With a suitable choice of q_1 ($0 < q_1 < \frac{1}{2m_1}$), the term $\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}\Phi_{m_1,q_1}(x_1;T_{c_1})$ can be made continuous. Even so, the term $\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}\sigma$, which produces theoretically infinite signals whenever the system solutions cross the axis $x_1 = 0$ unless $\sigma = 0$, is also present in the controller.

(ii) In fact, the stability analysis in [18] assumes implicitly that the system solutions do not cross the axis $x_1 = 0$ before $\sigma = 0$. However, this assumption does not hold in general. For instance, consider the cases $x_1(0) = 0$ and $x_2(0) \neq 0$, or $|x_1(0)| \approx 0$ and $x_1(0)x_2(0) \ll 0$.

Remark 3. A similar approach can be followed to design finite-time controller. The finite-time stability property can only be induced by using non-smooth functions, which would yield the same "singularity" problem in the controller. However, yet not canceling the singular term with the controller, the finite-time stability is preserved [20]. Unfortunately, to ensure predefined-time stability, the singular term must be canceled.

Although the controller (10) is not global, it will be helpful to state a sufficient condition for it to work. With this aim, consider the phase portrait of the closed-loop system in Fig. 1.

The regions labeled as A, B, C, D can be described as:

- $A = \left\{x \in \mathbb{R}^2 : x_1 > 0, x_2 \geq 0\right\}$
- $B = \left\{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0, x_1\sigma \geq 0\right\}$
- $C = \left\{x \in \mathbb{R}^2 : x_1 < 0, x_2 \leq 0\right\}$
- $D = \left\{x \in \mathbb{R}^2 : x_1 < 0, x_2 > 0, x_1\sigma \geq 0\right\}$

On region A, $\dot{x}_1 = x_2 \geq 0$ and $\dot{x}_2 = -\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}x_2 - \Phi_{m_2,q_2}(\sigma;T_{c_2}) \leq -\Phi_{m_2,q_2}(\sigma;T_{c_2}) < -\Phi_{m_2,q_2}(x_2;T_{c_2})$. Then, every solution starting on A enters B (without crossing the line $x_1 = 0$) in at most T_{c_2} time units.

On region B, it is clearly impossible to cross the line $x_1 = 0$ without crossing the manifold $\sigma = 0$. Moreover $\dot{\sigma} = -\Phi_{m_2,q_2}(\sigma;T_{c_2})$, hence, every solution starting on B will reach the manifold $\sigma = 0$ in predefined-time T_{c_2} (without crossing the line $x_1 = 0$), and will stay on it thereafter.

In fact, every solution starting on $A \cup B$ will reach the manifold $\sigma = 0$ in predefined-time T_{c_2} (without crossing the line $x_1 = 0$), and will stay on it thereafter. By symmetry, the same happens in the region $C \cup D$, which means that the controller (10) will work for every initial condition on

$$A \cup B \cup C \cup D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\right\}. \quad (11)$$

This above analysis is summarized in the following lemma.

Lemma 3. *For the system* (8) *closed-loop with the controller* (9)-(10)*, if the initial conditions of system satisfy* $x_1(0)\sigma(0) \geq 0$ *and* $x_1(0) \neq 0$*, then* $x_1(t) = 0$ *and* $x_2(t) = 0$ *for* $t > T_{c_1} + T_{c_2}$.

Although controller (10) is not global, the above result can be used to construct a global predefined-time stabilizing controller for system (8), exploiting the predefined-time feature.

To this end, a smooth manifold on the region $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\right\}$ will be constructed. We will consider smooth manifolds of the form

$$\sigma_0 = x_2 + cx_1 = 0, \qquad c > 0, \quad (12)$$

i.e., linear manifolds. Note that for this linear manifold to be in the region (11), it must be that

$$c \leq \frac{1}{m_1 q_1 T_{c_1}} \frac{\exp\left(|x_1|^{m_1 q_1}\right)}{|x_1|^{m_1 q_1}}.$$

To find such a c, let's minimize the right side of the above inequality.

Definition 6. Let $m \geq 1$, $0 < q < \frac{1}{m}$ and $T_c > 0$. The function $f_{m,q,T_c} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$f_{m,q,T_c}(s) = \frac{1}{mqT_c} \frac{\exp\left(s^{mq}\right)}{s^{mq}}. \quad (13)$$

Lemma 4. *Let* $m \geq 1$, $0 < q < \frac{1}{m}$ *and* $T_c > 0$*. Then,*

$$\min_{s \in \mathbb{R}_+} f_{m,q,T_c}(s) = f_{m,q,T_c}(1).$$

Proof. Note that

$$\frac{df_{m,q,T_c}(s)}{ds} = \frac{\exp\left(s^{mq}\right)}{T_c s^{mq+1}}\left[s^{mq} - 1\right].$$

It can be easily seen then that

$$\frac{df_{m,q,T_c}(s)}{ds} \begin{cases} < 0 & \text{if} \quad s < 1 \\ = 0 & \text{if} \quad s = 1 \\ > 0 & \text{if} \quad s > 1, \end{cases}$$

which implies that $\min_{s \in \mathbb{R}_+} f_{m,q,T_c}(s) = f_{m,q,T_c}(1)$. ☐

From Definition 6 and Lemma 4, a good candidate for the parameter c is

$$c = f_{m_1,q_1,T_{c_1}}(1) = \frac{\exp(1)}{m_1 q_1 T_{c_1}}.$$

With this selection, not only the linear manifold $\sigma_0 = 0$ (12) lies in the region (11), but is also close to the non-smooth manifold $\sigma = 0$ (9) near the origin (see Fig. 1).

Having constructed this linear manifold, a time-based switched predefined-time controller will be used. In the first stage, the controller will drive the system trajectories to the linear manifold (which is in the region $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\}$). In the second stage, the controller (10) will be used.

Definition 7. The *Heaviside step function*, denoted by H, is a discontinuous function defined as

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0. \end{cases} \tag{14}$$

With the definition of the function (14), the global predefined-time stabilizing controller for system (8) described before can be expressed as

$$u = [1 - H(t - T_{c_0})] u_0 + H(t - T_{c_0}) u_1,$$

where:

- $u_0 = -cx_2 - \Phi_{m_0,q_0}(\sigma_0; T_{c_0})$, with $m_0 \geq 1$, $0 < q_0 < \frac{1}{m_0}$ and T_{c_0}, drives the system trajectories to the linear manifold $\sigma_0 = 0$ in a predefined time T_{c_0}, and
- $u_1 = -\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1} x_2 - \Phi_{m_2,q_2}(\sigma; T_{c_2})$ is the controller (10), which stabilizes the system trajectories in a predefined time $T_{c_1} + T_{c_2}$ by Lemma 3.

IV. A Second-Order Predefined-Time Controller

Consider the following class of nonlinear systems

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta}, \end{aligned} \tag{15}$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a known nonlinear vector-valued function, $\boldsymbol{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is known nonlinear matrix-valued function, which is assumed to be invertible for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, and, $\boldsymbol{\Delta} \in \mathbb{R}^n$ is a bounded and unknown disturbance such that $||\boldsymbol{\Delta}|| \leq \delta$, with δ a known constant. The initial conditions of this system are $\boldsymbol{x}_1(0) = \boldsymbol{x}_{1,0}$ and $\boldsymbol{x}_2(0) = \boldsymbol{x}_{2,0}$.

For the system (15), the following theorem provides a controller that drives the variables $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to zero in predefined-time in spite of the disturbance $\boldsymbol{\Delta}$.

Theorem 1. *Given a time $T_c > 0$, consider the controller*

$$\begin{aligned} \boldsymbol{u}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{u}_0(\boldsymbol{x}_1, \boldsymbol{x}_2)\left[1 - H(t - T_{c_0})\right] + \\ \boldsymbol{u}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) H(t - T_{c_0}), \end{aligned} \tag{16}$$

with the terms $\boldsymbol{u}_0(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\boldsymbol{u}_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ defined as

$$\begin{cases} \boldsymbol{u}_0(\boldsymbol{x}_1, \boldsymbol{x}_2) &= -\boldsymbol{B}^{-1}(\boldsymbol{x}_1, \boldsymbol{x}_2)\Big[\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + c\boldsymbol{x}_2 \\ & \quad + \boldsymbol{\Phi}_{m_0,q_0}(\boldsymbol{\sigma}_0; T_{c_0}) + k\frac{\boldsymbol{\sigma}_0}{||\boldsymbol{\sigma}_0||}\Big] \\ \boldsymbol{\sigma}_0 &= \boldsymbol{x}_2 + c\boldsymbol{x}_1, \end{cases} \tag{17}$$

$$\begin{cases} \boldsymbol{u}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) &= -\boldsymbol{B}^{-1}(\boldsymbol{x}_1, \boldsymbol{x}_2)\Big[\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \\ & \quad \frac{\partial \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1})}{\partial \boldsymbol{x}_1}\boldsymbol{x}_2 + \boldsymbol{\Phi}_{m_2,q_2}(\boldsymbol{\sigma}_1; T_{c_2}) + k\frac{\boldsymbol{\sigma}_1}{||\boldsymbol{\sigma}_1||}\Big] \\ \boldsymbol{\sigma}_1 &= \boldsymbol{x}_2 + \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1}), \end{cases} \tag{18}$$

where $c = \frac{\exp(1)}{m_1 q_1 T_{c_1}}$, $m_0 \geq 1$, $m_1 \geq 1$, $m_2 \geq 1$, $0 < q_0 < \frac{1}{m_0}$, $0 < q_1 < \frac{1}{2m_1}$ and $0 < q_2 < \frac{1}{m_2}$, $T_{c_0} = \alpha_0 T_c$, $T_{c_1} = \alpha_1 T_c$ and $T_{c_2} = \alpha_2 T_c$, with $\alpha_0, \alpha_1, \alpha_2 > 0$, $\alpha_0 + \alpha_1 + \alpha_2 = 1$, and $k > \delta$. Then, the system (15) *closed-loop with the controller* (16) *is predefined-time stable with weak predefined time T_c.*

Proof. Note that system (15) can be written componentwise as

$$\begin{aligned} \dot{x}_{1,i} &= x_{2,i} \\ \dot{x}_{2,i} &= f_i(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{b}_i^T(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \Delta_i, \end{aligned}$$

for $i = 1, \ldots, n$, where $\boldsymbol{x}_1 = [x_{1,1} \ \ldots \ x_{1,n}]^T$, $\boldsymbol{x}_2 = [x_{2,1} \ \ldots \ x_{2,n}]^T$, $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) = [f_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \ \ldots \ f_n(\boldsymbol{x}_1, \boldsymbol{x}_2)]^T$, $\boldsymbol{B}^T(\boldsymbol{x}_1, \boldsymbol{x}_2) = [\boldsymbol{b}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \ \ldots \ \boldsymbol{b}_n(\boldsymbol{x}_1, \boldsymbol{x}_2)]$ and $\boldsymbol{\Delta} = [\Delta_1 \ \ldots \ \Delta_n]^T$. Furthermore, the componentwise expressions of the variables $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ are

$$\begin{aligned} \sigma_{0,i} &= x_{2,i} + cx_{1,i} \\ \sigma_{1,i} &= x_{2,i} + f_{m_1,q_1,T_{c_1}}(||\boldsymbol{x}_1||)x_{1,i}. \end{aligned}$$

A similar analysis to that of Lemma 3, yield that a sufficient condition for the controller (18) to work is $x_{1,i}(0)\sigma_{1,i}(0) \geq 0$ and $x_{1,i} \neq 0$. Then, the selection of c is justified by Lemma 4.

For $0 \leq t \leq T_{c_0}$, the derivative of $\boldsymbol{\sigma}_0$ (17) is

$$\begin{aligned} \dot{\boldsymbol{\sigma}}_0 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta} + c\boldsymbol{x}_2 \\ &= -k\frac{\boldsymbol{\sigma}_0}{||\boldsymbol{\sigma}_0||} - \boldsymbol{\Phi}_{m_0,q_0}(\boldsymbol{\sigma}_0; T_{c_0}) + \boldsymbol{\Delta}. \end{aligned}$$

Thus, applying Lemma 2, $\boldsymbol{\sigma}_0 = 0$ is weakly predefined-time stable with weak predefined time T_{c_0}. This is, $\boldsymbol{\sigma}_0(t) = 0$ for $t \geq T_{c_0}$.

Now, for $t > T_{c_0}$, the derivative of $\boldsymbol{\sigma}_1$ (18) is

$$\begin{aligned} \dot{\boldsymbol{\sigma}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta} + \frac{\partial \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1})}{\partial \boldsymbol{x}_1}\boldsymbol{x}_2 \\ &= -k\frac{\boldsymbol{\sigma}_1}{||\boldsymbol{\sigma}_1||} - \boldsymbol{\Phi}_{m_2,q_2}(\boldsymbol{\sigma}_1; T_{c_2}) + \boldsymbol{\Delta}. \end{aligned}$$

Hence, applying Lemma 2, $\boldsymbol{\sigma}_1 = 0$ is weakly predefined-time stable with weak predefined time T_{c_2}. This is, $\boldsymbol{\sigma}_1(t) = 0$ for $t \geq T_{c_0} + T_{c_2}$.

Finally, for $t > T_{c_0} + T_{c_2}$, since $\boldsymbol{\sigma}_1 = 0$,

$$\dot{\boldsymbol{x}}_1 = -\boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1}),$$

and applying Lemma 1, $\boldsymbol{x}_1(t) = 0$ for $t > T_{c_0} + T_{c_1} + T_{c_2} = T_c$. Also note that $\boldsymbol{x}_2(t) = 0$ for $t > T_c$. Then, the origin of the system (15) closed-loop with (16) is weakly predefined-time stable with weak predefined time T_c. □

V. Application: Predefined-Time Tracking of Full-Actuated Mechanical Systems

A generic model of fully actuated mechanical systems of n degrees of freedom has the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{P}(\dot{\boldsymbol{q}}) + \boldsymbol{\gamma}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{19}$$

where $\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}} \in \mathbb{R}^n$ are the position, velocity and acceleration vectors in joint space; $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effects matrix, $\boldsymbol{P}(\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction and $\boldsymbol{\gamma}(\boldsymbol{q}) \in \mathbb{R}^n$ is the gravity effects vector.

Defining the variables $\boldsymbol{x}_1 = \boldsymbol{q}$, $\boldsymbol{x}_2 = \dot{\boldsymbol{q}}$ and $\boldsymbol{u} = \boldsymbol{\tau}$, the mechanical model (19) can be rewritten in the state-space form (15), where $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\boldsymbol{M}^{-1}(\boldsymbol{x}_1)\left[\boldsymbol{C}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_2 + \boldsymbol{P}(\boldsymbol{x}_2) + \boldsymbol{\gamma}(\boldsymbol{x}_1)\right]$ and $\boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{M}^{-1}(\boldsymbol{x}_1)$.

A common problem in mechanical systems control is to track a desired time-dependent trajectory described by the triplet $(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t))$ of desired position $\boldsymbol{q}_d(t) = [q_{d_1}(t) \quad \cdots \quad q_{d_n}(t)]^T \in \mathbb{R}^n$, velocity $\dot{\boldsymbol{q}}_d(t) = [\dot{q}_{d_1}(t) \quad \cdots \quad \dot{q}_{d_n}(t)]^T \in \mathbb{R}^n$ and acceleration $\ddot{\boldsymbol{q}}_d(t) = [\ddot{q}_{d_1}(t) \quad \cdots \quad \ddot{q}_{d_n}(t)]^T \in \mathbb{R}^n$, which are all assumed to be known.

To be consequent with the state space notation, the desired position and velocity vectors are redefined as $\boldsymbol{x}_{1,d} = \boldsymbol{q}_d$ and $\boldsymbol{x}_{2,d} = \dot{\boldsymbol{q}}_d = \dot{\boldsymbol{x}}_{1,d}$, respectively. Then, defining the error variables as $\boldsymbol{e}_1 = \boldsymbol{x}_1 - \boldsymbol{x}_{1,d}$ (position error) and $\boldsymbol{e}_2 = \boldsymbol{x}_2 - \boldsymbol{x}_{2,d}$ (velocity error), the error dynamics are:

$$\begin{aligned} \dot{\boldsymbol{e}}_1 &= \boldsymbol{e}_2 \\ \dot{\boldsymbol{e}}_2 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} - \ddot{\boldsymbol{x}}_{1,d}. \end{aligned} \tag{20}$$

The error variables $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are to be stabilized in predefined time with available measurements of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{1,d}, \boldsymbol{x}_{2,d} = \dot{\boldsymbol{x}}_{1,d}$ and $\ddot{\boldsymbol{x}}_{1,d}$. To this end, the controller in Theorem 1 is used.

VI. Example: Trajectory Tracking for a Two-Link Manipulator

A. Model description

Consider a planar, two-link manipulator with revolute joints as the one exposed in Example 12.1 of [21]. The manipulator link lengths are L_1 and L_2, the link masses (concentrated in the end of each link) are M_1 and M_2. The manipulator is operated in the plane, such that the gravity acts along the $z-$axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass M_2 is concentrated) position (x_w, y_w) is given by $x_w = L_1\cos(q_1) + L_2\cos(q_1 + q_2)$ and $y_w = L_1\sin(q_1) + L_2\sin(q_1 + q_2)$, where q_1 and q_2 are the joint positions (angular positions).

Applying the Euler-Lagrange equations, a model according to (19) is obtained, with $m_{11} = L_1^2(M_1 + M_2) + 2(L_2^2 M_2 + L_1 L_1 M_2 \cos q_2) - L_2^2 M_2$, $m_{12} = m_{21} = L_2^2 M_2 + L_1 L_1 M_2 \cos q_2$, $m_{22} = L_2^2 M_2$, $h = L_1 L_2 M_2 \sin q_2$, $c_{11} = -h\dot{q}_2$, $c_{12} = -h(\dot{q}_1 + \dot{q}_2)$, $c_{21} = h\dot{q}_1 \quad c_{22} = 0$, and

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$
$$\boldsymbol{P}(\dot{\boldsymbol{q}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\gamma}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius r_d and center in the origin.

The two-link manipulator parameters are $M_1 = M_2 = 0.2\,\mathrm{kg}$ and $L_1 = L_2 = 0.2\,\mathrm{m}$.

B. Simulation results

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The initial conditions for the two-link manipulator were selected as: $x_1(0) = \left[-\frac{3\pi}{4} \quad -\frac{\pi}{4}\right]^T$ and $x_2(0) = [0 \quad 0]^T$. In addition, the controller gains were adjusted to: $k = 0$, $T_{c_0} = T_{c_1} = 1$, $T_{c_2} = 0.1$, $m_0 = m_1 = m_2 = 1$, $q_0 = q_2 = \frac{1}{2}$ and $q_1 = 0.3$.

The desired circular trajectory in the joint coordinates is described by the equations

$$q_d(t) = x_{1,d}(t) = \begin{bmatrix} q_{d_1}(t) \\ q_{d_2}(t) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2}t - \pi \\ -\frac{\pi}{2} \end{bmatrix},$$

and it corresponds to a circumference of radius $0.2828\,\mathrm{m}$.

The following figures show the behavior of the proposed controller.

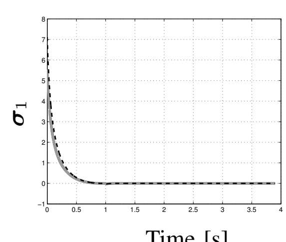

Fig. 2: Variable $\boldsymbol{\sigma}_1$. First component (gray and solid) and second component (black and dashed). Note that $\boldsymbol{\sigma}_1(t) = 0$ for $t > T_{c_2} = 1.1$ s.

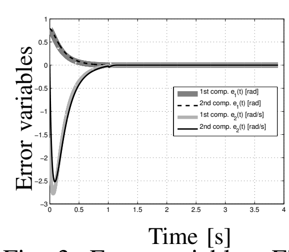

Fig. 3: Error variables. First component of e_1 (dark gray and thick), second component of e_1 (black and dashed), first component of e_2 (light gray and solid) and second component of e_2 (black and solid).

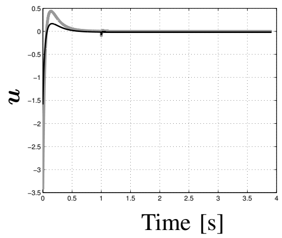

Fig. 4: Control signal. First component (gray and solid) and second component (black and solid).

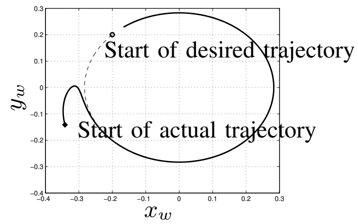

Fig. 5: Actual trajectory (x_w, y_w) (black and solid) and desired trajectory $(x_{w,d}, y_{w,d})$ (black and dashed).

Note that $\boldsymbol{\sigma}_1(t) = 0$ for $t \geq 1.1$ s $= T_{c_0} + T_{c_2}$ (Fig. 2). Once the error variables slide over the manifold $\boldsymbol{\sigma}_1 = 0$, this motion is governed by the reduced order system

$$\dot{e}_1 = e_2 = -\Phi_{p_1}(e_1; T_{c_1}).$$

This imply that the error variables are exactly zero for $t > T_{c_0} + T_{c_1} + T_{c_2} = 2.1$ s. In fact, from Fig. 3, it can be seen that $e_1(t) = e_2 = 0$ for $t \geq 1.5$ s $< T_{c_0} + T_{c_1} + T_{c_2} = 2.1$ s. Fig. 4 shows the control signal (torque) versus time, where the switching effect can be seen at $t = 2.1$ s $= T_{c_0}$. Finally, from Fig. 5, it can be seen the reference tracking in rectangular coordinates.

VII. Conclusion

The predefined-time stabilization of second-order systems was revisited in this paper. The region where the controller developed in [18] does not undergo singularities was estimated. Moreover, a time-based switched controller is proposed to first drive the system trajectories to the estimated region in predefined time and then uses the controller in [18]. This controller was applied to the trajectory tracking control in fully actuated mechanical systems. An illustrative example of the control of a two-link planar manipulator with predefined-time convergence showed the effectiveness of the proposed algorithm.

Acknowledgment

Esteban Jiménez acknowledges to CONACyT, México for the MSc scholarship number 426598.

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