A Note on Predefined-Time Stability [⋆]

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Abstract: This note presents a new characterization for predefined-time-stable systems based on Lyapunov stability. In contrast to the previous results in predefined-time stability, the proposed characterization allows the construction of predefined-time stabilizing controllers with polynomial terms instead of exponential ones, removing the exponential nature hypothesis of predefined-time-stable systems. Moreover, the existing Lyapunov characterization of predefined-time stability is shown to be a consequence of the new theorem presented in this paper. Finally, the proposed approach is used for the construction of robust predefined-time stabilizing controllers for first-order systems. A simulation example shows the feasibility of the proposed methods.

Keywords: Predefined-time stability, Fixed-time stability, Sliding mode control, Nonlinear systems, Lyapunov's direct method.

1. INTRODUCTION

The concept of *finite-time stability* addresses solutions to several different applications that are characterized for requiring hard time-response constraints. Some important works on this topic and its applications to control systems have been carried out in Roxin (1966); Haimo (1986); Utkin (1992); Bhat and Bernstein (2000); Moulay and Perruquetti (2008). However, generally this finite time is an unbounded (not uniformly globally bounded) function of the initial conditions of the system. A desired feature is to eliminate this boundlessness condition, for instance in estimation or optimization problems. This gives rise to a stronger form of stability called *fixed-time stability*, where the settling-time function is bounded. The notion of fixed-time stability have been investigated in Andrieu et al. (2008); Cruz-Zavala et al. (2010); Polyakov (2012); Fraguela et al. (2012); Polyakov and Fridman (2014).

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time. To overcome the above, it is necessary to consider another class of dynamical systems that exhibit the property of *predefined-time stability*, which have been studied in (Sánchez-Torres et al., 2014, 2015, 2018; Jiménez-Rodríguez et al., 2017a,c,b). For these systems, an upper bound (sometimes the least upper bound) of the fixed stabilization time appears explicitly in their tuning gains.

Along this way, the Lyapunov methods have proved to be a very effective tool for analyzing and designing nonlinear control systems (Bacciotti and Rosier, 2005). In the same manner, they have been highly used for convergence rate estimation in systems exhibiting the finite-time and fixed-time stability property (Bhat and Bernstein, 2000; Polyakov, 2012; Defoort et al., 2016; Zuo et al., 2018), and in particular in systems with sliding modes (Polyakov and Fridman, 2014). For systems exhibiting the predefined-time stability property, this has not been different. Sánchez-Torres et al. (2018) proposed a Lyapunov-like theorem for predefined-time stability. However, using this Lyapunov-like condition for controller design, only predefined-time stabilizing controllers with exponential terms were obtained (Jiménez-Rodríguez et al., 2017a,b; Sánchez-Torres et al., 2018).

In this note, a new Lyapunov-like theorem for predefined-time stability is stated. This theorem allows the construction of predefined-time stabilizing controllers with polynomial terms instead of exponential ones, removing the exponential nature hypothesis of predefined-time-stable systems. Moreover, the Lyapunov-like theorem presented by Sánchez-Torres et al. (2018) is shown to be a direct consequence of the results developed in this paper.

2. PRELIMINARIES

2.1 On finite-time, fixed-time and predefined-time stability

Consider the system

$$\dot{x} = f(x; \rho), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, $\boldsymbol{\rho} \in \mathbb{R}^b$ represents the parameters of the system, $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, and $\boldsymbol{f}(\boldsymbol{0}; \boldsymbol{\rho}) = \boldsymbol{0}$. The initial conditions of this system are $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \mathbb{R}^n$.

Definition 1. (Bhat and Bernstein (2000)). The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(\boldsymbol{x}_0) : \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{0}$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the *settling-time function*.

Definition 2. (Polyakov (2012)). The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall \boldsymbol{x}_0 \in \mathbb{R}^n : T(\boldsymbol{x}_0) \leq T_{\max}$.

Remark 3. Assuming that the origin of (1) is fixed-time stable, the bound $T_{\max}$ in Definition 2 is trivially non-unique; for instance, note that $T(\boldsymbol{x}_0) \leq \lambda T_{\max}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition 4. (Sánchez-Torres et al. (2014)). Let the origin be fixed-time-stable for the system (1). The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \{T_{\max} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\max}, \ \forall \boldsymbol{x}_0 \in \mathbb{R}^n\}.$$

In addition, the least upper bound of the settling-time function, denoted by T_f, is defined as

$$T_f = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0).$$

Remark 5. For some applications such as state estimation, dynamic optimization, fault detection, among others, it would be convenient that the trajectories of the system (1) reach the origin within a time $T_c \in \mathcal{T}$, which can be defined in advance as function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. This desirable property motivates the definition of predefined-time stability.

Definition 6. (Sánchez-Torres et al. (2018)). For the system parameters $\boldsymbol{\rho}$ and a constant $T_c := T_c(\boldsymbol{\rho}) > 0$, the origin of (1) is said to be *predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

If this is the case, T_c is called a *predefined time*.

Remark 7. It would be desirable to choose $T_c = T_c(\boldsymbol{\rho})$ not only as a bound of the settling-time function $T_c \in \mathcal{T}$, but as the least upper bound, i.e., $T_c = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0)$. However, this selection requires complete knowledge about the system, compromising its application to uncertain systems.

2.2 A vector power function and some properties

In the foregoing, $||\cdot||$ will stand for the euclidean 2-norm in $\mathbb{R}^n$.

Definition 8. (Jiménez-Rodríguez et al. (2018)). Let $h \geq 0$. For $\boldsymbol{x} \in \mathbb{R}^n$, define the function

$$||\lfloor \boldsymbol{x} \rceil||^h = \frac{\boldsymbol{x}}{||\boldsymbol{x}||^{1-h}}, \tag{2}$$

with $||\boldsymbol{x}||$ the euclidean norm of $\boldsymbol{x}$. Since $\lim_{\boldsymbol{x} \to 0} ||\lfloor \boldsymbol{x} \rceil||^h = \boldsymbol{0}$ for $h > 0$, it is considered that $||\lfloor \boldsymbol{0} \rceil||^h = \boldsymbol{0}$. Therefore, the function $||\lfloor \boldsymbol{x} \rceil||^h$ is continuous for $0 \leq h < 1$ and discontinuous in $\boldsymbol{x} = \boldsymbol{0}$ for $h = 0$.

Proposition 9. (Jiménez-Rodríguez et al. (2018)). For $h > 0$, the function $||\lfloor \boldsymbol{x} \rceil||^h$ fulfills:

(i) $||\lfloor -\boldsymbol{x} \rceil||^h = -||\lfloor \boldsymbol{x} \rceil||^h$.

(ii) $||\lfloor \boldsymbol{x} \rceil||^0 = \frac{\boldsymbol{x}}{||\boldsymbol{x}||}$, a unit vector.

(iii) $||\lfloor \boldsymbol{x} \rceil||^1 = ||\lfloor \boldsymbol{x} \rceil|| = \boldsymbol{x}$.

(iv) $\frac{\partial ||\lfloor \boldsymbol{x} \rceil||^h}{\partial \boldsymbol{x}} = \left[\boldsymbol{I}_n + (h-1) \frac{\boldsymbol{x} \boldsymbol{x}^T}{||\boldsymbol{x}||^2} \right] ||\boldsymbol{x}||^{h-1}$ and $\frac{\partial ||\boldsymbol{x}||^h}{\partial \boldsymbol{x}} = h ||\lfloor \boldsymbol{x}^T \rceil||^{h-1}$, where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

(v) For $h_1, h_2 \in \mathbb{R}$, it follows:

    · $||\boldsymbol{x}||^{h_1} ||\boldsymbol{x}||^{h_2} = ||\boldsymbol{x}||^{h_1+h_2}$,

    · $||\lfloor \boldsymbol{x} \rceil||^{h_1} ||\boldsymbol{x}||^{h_2} = ||\boldsymbol{x}||^{h_1} ||\lfloor \boldsymbol{x} \rceil||^{h_2} = ||\lfloor \boldsymbol{x} \rceil||^{h_1+h_2}$, and

    · $||\lfloor \boldsymbol{x}^T \rceil||^{h_1} ||\lfloor \boldsymbol{x} \rceil||^{h_2} = ||\boldsymbol{x}||^{h_1+h_2}$.

(vi) If $h_1, h_2 > 0$, then $\left| \left| \lfloor ||\lfloor \boldsymbol{x} \rceil||^{h_1} \rceil \right| \right|^{h_2} = ||\lfloor \boldsymbol{x} \rceil||^{h_1 h_2}$.

3. A LYAPUNOV-LIKE CHARACTERIZATION OF PREDEFINED-TIME STABILITY

In this section, a new Lyapunov-like theorem for predefined-time stability is stated. This theorem allows the construction of predefined-time stabilizing terms *without an exponential term* (see Section 4, Examples 20 and 21), removing the exponential nature hypothesis of predefined-time-stable systems. Moreover, the Lyapunov-like theorem presented by Sánchez-Torres et al. (2015, 2018) is shown to be a consequence in Corollary 12.

For all of the above, the following theorem constitutes the main contribution of this paper.

Theorem 10. If there exists a continuous function $W : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &0 \leq W(\boldsymbol{x}) < 1, \\ &W(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0, \text{ and} \\ &\lim_{||\boldsymbol{x}|| \to \infty} W(\boldsymbol{x}) = 1, \end{aligned}$$

and any solution $\boldsymbol{x}(t)$ of (1) satisfies

$$\dot{W}(\boldsymbol{x}) \leq -\frac{1}{T_c}, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \tag{3}$$

for a constant $T_c := T_c(\boldsymbol{\rho}) > 0$, then the system (1) origin is a predefined-time-stable equilibrium point with predefined time T_c.

Proof. In virtue of $0 \leq W(\boldsymbol{x}) < 1$, $W(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0$, $\lim_{||\boldsymbol{x}|| \to \infty} W(\boldsymbol{x}) = 1$, and the inequality (3), one has that $V(\boldsymbol{x}) = -\log(1 - W(\boldsymbol{x}))$ is a radially unbounded Lyapunov function for system (1). Consequently, the system (1) origin is a globally finite-time-stable equilibrium point. From (3) and $W(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0$, the settling-time function complies to

$$T(\boldsymbol{x}_0) \leq T_c W(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

Since $T_c = T_c(\boldsymbol{\rho})$ is an upper bound for the settling-time function, the origin of the system (1) is in fact predefined-time-stable with predefined time T_c.

Theorem 10 has the following immediate corollary.

Corollary 11. Under the same conditions of Theorem 10, if for any solution $\boldsymbol{x}(t)$ of (1)

$$\dot{W}(\boldsymbol{x}) = -\frac{1}{T_c}, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$$

holds, then

$$\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c.$$

Proof. Since the equality version of (3) holds, the settling-time function is exactly

$$T(\boldsymbol{x}_0) = T_c W(\boldsymbol{x}_0).$$

Furthermore,

$$\sup_{x_0 \in \mathbb{R}^n} W(\boldsymbol{x}_0) = \lim_{||x_0|| \to \infty} W(\boldsymbol{x}_0) = 1,$$

and the result is obtained.

Furthermore, Theorem 2.1 of Sánchez-Torres et al. (2015, 2018) is shown to be a consequence of Theorem 10 in the following corollary.

Corollary 12. ((Sánchez-Torres et al., 2015, 2018)). If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(\boldsymbol{x}) \geq 0 \text{ and}$$
$$V(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0,$$

and any solution $\boldsymbol{x}(t)$ of (1) satisfies

$$\dot{V}(\boldsymbol{x}) \leq -\frac{1}{pT_c} \exp\left(V(\boldsymbol{x})^p\right) V(\boldsymbol{x})^{1-p}, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}, \tag{4}$$

for constants $T_c := T_c(\boldsymbol{\rho}) > 0$ and $0 < p \leq 1$, then the origin (1) is predefined-time-stable for system (1) whose predefined time is T_c.

Proof. Consider the function $W(\boldsymbol{x}) = 1 - \exp(-V(\boldsymbol{x})^p)$. Since V is a positive definite function, it results $0 \leq W(\boldsymbol{x}) < 1$.

Furthermore, $W(\boldsymbol{x}) = 0 \Leftrightarrow 1 - \exp(-V(\boldsymbol{x})^p) = 0 \Leftrightarrow \exp(-V(\boldsymbol{x})^p) = 1 \Leftrightarrow V(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0$.

Finally, noticing that $\frac{dW}{dV} = p \exp(-V(\boldsymbol{x})^p) V(\boldsymbol{x})^{p-1} \geq 0$, in virtue of inequality (4), $\dot{V}(\boldsymbol{x}) \leq -\frac{1}{T_c} \left[\frac{dW}{dV}\right]^{-1}$, then, using the chain rule it yields

$$\dot{W}(\boldsymbol{x}) = \frac{dW}{dV} \dot{V}(\boldsymbol{x}) \leq -\frac{1}{T_c}.$$

Since all the hypotheses of Theorem 10 hold, the origin of system (1) is predefined-time-stable, with a predefined time T_c.

4. LYAPUNOV-BASED PREDEFINED-TIME-STABLE SYSTEMS CONSTRUCTION

4.1 Basics

Consider the unperturbed first-order dynamical system

$$\dot{\boldsymbol{x}} = -k\boldsymbol{\phi}(\boldsymbol{x}), \tag{5}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ the system state, $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ is a non-smooth function of the state to be constructed such that the origin is a predefined-time-stable equilibrium of (5), and $k > 0$ is a parameter.

For the construction of the function $\boldsymbol{\phi}(\boldsymbol{x})$ let's assume that a candidate function $W(\boldsymbol{x})$ which complies to $0 \leq W(\boldsymbol{x}) < 1$, $W(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0$, $\lim_{||\boldsymbol{x}|| \to \infty} W(\boldsymbol{x}) = 1$, and it is C^1 (continuously differentiable) in $\mathbb{R}^n \setminus \{0\}$, has been encountered. Then,

$$\dot{W}(\boldsymbol{x}) = \frac{\partial W}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = -k \frac{\partial W}{\partial \boldsymbol{x}} \boldsymbol{\phi}(\boldsymbol{x}),$$

where $\frac{\partial W}{\partial \boldsymbol{x}}^T = \begin{bmatrix} \frac{\partial W}{\partial x_1} & \cdots & \frac{\partial W}{\partial x_n} \end{bmatrix}^T \in \mathbb{R}^n$ is the gradient of the scalar function $W(\boldsymbol{x})$.

Assuming $\left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\| \neq 0$, which is equivalent to assume that rank $\left(\frac{\partial W}{\partial \boldsymbol{x}}\right) = 1$ or that $\frac{\partial W}{\partial \boldsymbol{x}}$ is full rank, the function $\boldsymbol{\phi}(\boldsymbol{x})$ and the parameter k can be designed as

$$\boldsymbol{\phi}(\boldsymbol{x}) = \left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}}\right]^T, \quad k = \frac{1}{T_c},$$

to achieve (see Theorem 10 and Corollary 11)

$$\dot{W}(\boldsymbol{x}) = -\frac{1}{T_c}.$$

Remark 13. The assumption $\frac{\partial W}{\partial \boldsymbol{x}}$ is full rank implies that it is right invertible (seeing it as an $1 \times n$ matrix). The right inverse which provides minimum norm solution is then

$$\frac{\partial W}{\partial \boldsymbol{x}}^T \left(\frac{\partial W}{\partial \boldsymbol{x}} \frac{\partial W}{\partial \boldsymbol{x}}^T\right)^{-1} = \frac{\partial W}{\partial \boldsymbol{x}}^T \left(\left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\|^2\right)^{-1},$$

that is in fact, the selection of $\boldsymbol{\phi}(\boldsymbol{x})$.

The above construction is summarized in the following lemma.

Lemma 14. Assume there is a continuous function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying

  (i) $0 \leq W(\boldsymbol{x}) < 1$ for all $x \in \mathbb{R}^n$,
  (ii) $W(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$,
  (iii) $W(\boldsymbol{x}) \to 1$ as $||\boldsymbol{x}|| \to \infty$,
  (iv) $W(\boldsymbol{x})$ is C^1 in $\mathbb{R}^n \setminus \{0\}$, and
  (v) $\left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\| \neq 0$ for all $x \in \mathbb{R}^n$.

Then, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{1}{T_c} \left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}}\right]^T,$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$ (see Corollary 11).

4.2 Function $W(\boldsymbol{x})$ design

From Lemma 14 it follows that for each function $W : \mathbb{R}^n \to \mathbb{R}$ complying *(i), (ii), (iii), (iv)* and *(v)*, a dynamical system with predefined-time-stable origin can be constructed.

In the following, some examples of functions $W(\boldsymbol{x})$ which fulfills the conditions *(i), (ii), (iii), (iv)* and *(v)* of Lemma 14 are presented. The first of them generalizes the system proposed by Sánchez-Torres et al. (2018) to a larger class of systems with exponential functions. For this case, $W(\boldsymbol{x})$ recalls the response of a stable first order linear filter to a step input.

Example 15. Consider the function

$$W(\boldsymbol{x}) = 1 - a^{-||\boldsymbol{x}||^q},$$

with the parameters $a > 1$, and $0 < q \leq 1$. Note that:

  (i) Since $0 < a^{-||\boldsymbol{x}||^q} \leq 1$, the function $W(\boldsymbol{x})$ complies to $0 \leq W(\boldsymbol{x}) < 1$.
  (ii) Clearly, $W(\boldsymbol{x}) = 1 - a^{-||\boldsymbol{x}||^q} = 0$ if and only if $\boldsymbol{x} = 0$.
  (iii) Since $a^{-||\boldsymbol{x}||^q} \to 0$ as $||\boldsymbol{x}|| \to \infty$, the function $W(\boldsymbol{x}) = 1 - a^{-||\boldsymbol{x}||^q} \to 1$ as $||\boldsymbol{x}|| \to \infty$.
  (iv) $\frac{\partial W}{\partial \boldsymbol{x}} = q \ln(a) a^{-||\boldsymbol{x}||^q} \lfloor \boldsymbol{x}^T \rceil^{q-1}$ exists and is continuous on $\mathbb{R}^n \setminus \{0\}$.
  (v) $\left\|\frac{\partial W}{\partial \boldsymbol{x}}\right\| = q \ln(a) a^{-||\boldsymbol{x}||^q} ||\boldsymbol{x}||^{q-1} \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Then, using Lemma 14, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{1}{q \ln(a) T_c} a^{-||\boldsymbol{x}||^q} \lfloor \boldsymbol{x} \rceil^{1-q}$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

For $a = e = \exp(1)$, the system (*) reduces to the system proposed in Sánchez-Torres et al. (2018).

Remark 16. Every continuous cumulative distribution function fulfills condition *(i)* of Lemma 14. Then, good candidates for function $W(\boldsymbol{x})$ can be obtained taking a look after continuous cumulative distribution functions. The following example considers a standard normal distribution-based function.

Example 17. Consider the function

$$W(\boldsymbol{x}) = \mathrm{erf}\left(\frac{||\boldsymbol{x}||^q}{\sqrt{2}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^{||\boldsymbol{x}||^q} \exp\left(-\frac{\xi^2}{2}\right) \mathrm{d}\xi,$$

with $0 < q \leq 1$ a parameter. The function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\mathrm{d}t$ stands for the error function (Abromowitz and Stegun, 1965). Note that:

(i) $||\boldsymbol{x}||^q \geq 0$, then $0 \leq W(\boldsymbol{x}) = \mathrm{erf}\left(\frac{||\boldsymbol{x}||^q}{\sqrt{2}}\right) < 1$.

(ii) $W(\boldsymbol{x}) = \mathrm{erf}\left(\frac{||\boldsymbol{x}||^q}{\sqrt{2}}\right)$ if and only if $\boldsymbol{x} = 0$.

(iii) The function $W(\boldsymbol{x}) = \mathrm{erf}\left(\frac{||\boldsymbol{x}||^q}{\sqrt{2}}\right) \to 1$ as $||\boldsymbol{x}|| \to \infty$.

(iv) $\frac{\partial W}{\partial \boldsymbol{x}} = \frac{2q}{\sqrt{2\pi}} \exp\left(-\frac{||\boldsymbol{x}||^{2q}}{2}\right) \lfloor \boldsymbol{x}^T \rceil^{q-1}$ exists and is continuous on $\mathbb{R}^n \setminus \{0\}$.

(v) $||\frac{\partial W}{\partial \boldsymbol{x}}|| = \frac{2q}{\sqrt{2\pi}} \exp\left(-\frac{||\boldsymbol{x}||^{2q}}{2}\right) ||\boldsymbol{x}||^{q-1} \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Then, using Lemma 14, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{\sqrt{2\pi}}{2qT_c} \exp\left(\frac{||\boldsymbol{x}||^{2q}}{2}\right) \lfloor \boldsymbol{x} \rceil^{1-q}$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Remark 18. The condition *(i)* of Lemma 14 is related to the right side of odd symmetric sigmoid functions. Thus, $W(\boldsymbol{x})$ can be designed with appropriate scaling and translations of well-known sigmoid functions. The following examples are based on those functions.

Example 19. Consider the function

$$W(\boldsymbol{x}) = \frac{1 - a^{-||\boldsymbol{x}||^q}}{1 + b^{-||\boldsymbol{x}||^p}},$$

with the parameters $a, b > 1$, and $0 < p, q \leq 1$. Note that:

(i) Since $0 < a^{-||\boldsymbol{x}||^q} \leq 1$ and $0 < b^{-||\boldsymbol{x}||^p} \leq 1$, the function $W(\boldsymbol{x})$ complies to $0 \leq W(\boldsymbol{x}) < 1$.

(ii) Clearly, $W(\boldsymbol{x}) = \frac{1 - a^{-||\boldsymbol{x}||^q}}{1 + b^{-||\boldsymbol{x}||^p}} = 0$ if and only if $\boldsymbol{x} = 0$.

(iii) Since $a^{-||\boldsymbol{x}||^q}, b^{-||\boldsymbol{x}||^p} \to 0$ as $||\boldsymbol{x}|| \to \infty$, the function $W(\boldsymbol{x}) \to 1$ as $||\boldsymbol{x}|| \to \infty$.

(iv) $\frac{\partial W}{\partial \boldsymbol{x}} = \frac{q \ln(a) a^{-||\boldsymbol{x}||^q} \lfloor \boldsymbol{x}^T \rceil^{q-1}}{1 + b^{-||\boldsymbol{x}||^p}} + \frac{p \ln(b) b^{||\boldsymbol{x}||^p} \left(1 - a^{-||\boldsymbol{x}||^q}\right) \lfloor \boldsymbol{x}^T \rceil^{p-1}}{\left(1 + b^{||\boldsymbol{x}||^p}\right)^2}$ exists and is continuous on $\mathbb{R}^n \setminus \{0\}$.

(v) $||\frac{\partial W}{\partial \boldsymbol{x}}|| \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Then, using Lemma 14, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{\left(1 + b^{-||\boldsymbol{x}||^p}\right)\left(1 + b^{||\boldsymbol{x}||^p}\right)^2}{T_c} \frac{\varphi(\boldsymbol{x})}{||\varphi(\boldsymbol{x})||^2}$$

with $\varphi(\boldsymbol{x}) = q \ln(a) a^{-||\boldsymbol{x}||^q} \left(1 + b^{||\boldsymbol{x}||^p}\right)^2 \lfloor \boldsymbol{x} \rceil^{q-1} + p \ln(b) b^{||\boldsymbol{x}||^p} \left(1 - a^{-||\boldsymbol{x}||^q}\right) \left(1 + b^{-||\boldsymbol{x}||^p}\right) \lfloor \boldsymbol{x} \rceil^{p-1}$ is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

For $a = b = e^{2\alpha} = \exp(2\alpha)$ with $\alpha > 0$, and $p = q$, it follows $W(\boldsymbol{x}) = \tanh(\alpha ||\boldsymbol{x}||^q)$. Then, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{1}{\alpha q T_c} \cosh^2(\alpha ||\boldsymbol{x}||^q) \lfloor \boldsymbol{x} \rceil^{1-q}$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Example 20. Consider the function

$$W(\boldsymbol{x}) = \frac{||\boldsymbol{x}||^q}{||\boldsymbol{x}||^q + 1},$$

with $0 < q \leq 1$ a parameter. Note that:

(i) $||\boldsymbol{x}||^q \geq 0$, then $0 \leq W(\boldsymbol{x}) = \frac{||\boldsymbol{x}||^q}{||\boldsymbol{x}||^q + 1} < \frac{||\boldsymbol{x}||^q}{||\boldsymbol{x}||^q} = 1$.

(ii) $\frac{||\boldsymbol{x}||^q}{||\boldsymbol{x}||^q + 1} = 0$ if and only if $||\boldsymbol{x}||^q = 0$, if and only if $\boldsymbol{x} = 0$.

(iii) Clearly, the function $W(\boldsymbol{x}) = \frac{||\boldsymbol{x}||^q}{||\boldsymbol{x}||^q + 1} \to 1$ as $||\boldsymbol{x}|| \to \infty$.

(iv) $\frac{\partial W}{\partial \boldsymbol{x}} = \frac{q}{(||\boldsymbol{x}||^q + 1)^2} \lfloor \boldsymbol{x}^T \rceil^{q-1}$ exists and is continuous on $\mathbb{R}^n \setminus \{0\}$.

(v) $||\frac{\partial W}{\partial \boldsymbol{x}}|| = \frac{q}{(||\boldsymbol{x}||^q + 1)^2} ||\boldsymbol{x}||^{q-1} \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Then, using Lemma 14, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{1}{q T_c} \left(||\boldsymbol{x}||^q + 1\right)^2 \lfloor \boldsymbol{x} \rceil^{1-q}$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Example 21. Consider the function

$$W(\boldsymbol{x}) = \frac{2}{\pi} \tan^{-1}(||\boldsymbol{x}||^q),$$

with $0 < q \leq 1$ a parameter. Note that:

(i) $||\boldsymbol{x}||^q \geq 0$, then $0 \leq W(\boldsymbol{x}) = \frac{2}{\pi} \tan^{-1}(||\boldsymbol{x}||^q) < 1$.

(ii) $\frac{2}{\pi} \tan^{-1}(||\boldsymbol{x}||^q) = 0$ if and only if $||\boldsymbol{x}||^q = 0$, if and only if $\boldsymbol{x} = 0$.

(iii) Since $\tan^{-1}(||\boldsymbol{x}||^q) \to \frac{\pi}{2}$ as $||\boldsymbol{x}|| \to \infty$, the function $W(\boldsymbol{x}) = \frac{2}{\pi} \tan^{-1}(||\boldsymbol{x}||^q) \to 1$ as $||\boldsymbol{x}|| \to \infty$.

(iv) $\frac{\partial W}{\partial \boldsymbol{x}} = \frac{2q}{\pi(||\boldsymbol{x}||^{2q} + 1)} \lfloor \boldsymbol{x}^T \rceil^{q-1}$ exists and is continuous on $\mathbb{R}^n \setminus \{0\}$.

(v) $||\frac{\partial W}{\partial \boldsymbol{x}}|| = \frac{2q}{\pi(||\boldsymbol{x}||^{2q} + 1)} ||\boldsymbol{x}||^{q-1} \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Then, using Lemma 14, the origin of the system

$$\dot{\boldsymbol{x}} = -\frac{\pi}{2q T_c} \left(||\boldsymbol{x}||^{2q} + 1\right) \lfloor \boldsymbol{x} \rceil^{1-q}$$

is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Remark 22. Unlike the dynamical system obtained in Example 15 (Sánchez-Torres et al., 2018), the systems obtained in Examples 20 and 21 contain polynomial terms and not exponential terms. This represents a significant advantage, for example in controller design since far from the origin the polynomial terms imply less control energy than exponential terms:

$$\lim_{||\boldsymbol{x}|| \to \infty} \frac{(||\boldsymbol{x}||^q + 1)^2}{\exp(||\boldsymbol{x}||^q)} = \lim_{||\boldsymbol{x}|| \to \infty} \frac{\pi}{2} \frac{||\boldsymbol{x}||^{2q} + 1}{\exp(||\boldsymbol{x}||^q)} = 0.$$

From the above, numerically more stable algorithms can also be expected.

Remark 23. The predefined-time-stable dynamical system constructed in Example 21, can be written as

$$\dot{\boldsymbol{x}} = -\frac{\pi}{2q T_c} \lfloor \boldsymbol{x} \rceil^{1+q} - \frac{\pi}{2q T_c} \lfloor \boldsymbol{x} \rceil^{1-q},$$

which is a typical form of fixed-time-stable systems. To see this, consider the continuous radially unbounded function $V(\boldsymbol{x}) = ||\boldsymbol{x}||$, whose time-derivative along the trajectories of the above system is

$$\dot{V}(\boldsymbol{x}) = -\frac{\pi}{2q T_c} (V(\boldsymbol{x})^{1+q} + V(\boldsymbol{x})^{1-q}).$$

Using Theorem 13 of Polyakov and Fridman (2014), one could assure at most that the origin of the system is fixed-time-stable with

$$T(\boldsymbol{x}_0) \leq \frac{4T_c}{\pi} \approx 1.27T_c,$$

leading to a 27% overestimation of the time of convergence.

5. FIRST-ORDER PREDEFINED-TIME CONTROLLERS

To apply the results in Section 4 to robust first-order controller design, consider the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{u} + \boldsymbol{\Delta}(t, \boldsymbol{x}) \qquad (6)$$

with $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{\Delta} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. The objective is to stabilize system (6) at the origin in a predefined time T_c, starting from an arbitrary initial state $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \mathbb{R}^n$ and in spite of the unknown perturbation $\boldsymbol{\Delta}(t, \boldsymbol{x})$.

Remark 24. The control system (6) does not stand for an overall perturbed system. It instead represents a sliding variable dynamics with (partial) equivalent control compensation Utkin (1992).

5.1 Direct rejection of the perturbation term

Theorem 25. Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a vanishing perturbation term such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq \delta ||\boldsymbol{x}||$, with $0 < \delta < \infty$ a known constant, and assume there exists a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying *(i), (ii), (iii), (iv)* and *(v)* of Lemma 14. Selecting the control input as

$$\boldsymbol{u} = -\frac{1}{T_c} \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}} \right]^T - k\boldsymbol{x}, \qquad (7)$$

with $T_c > 0$ and $k \geq \delta$, the origin of system (6) closed by (7) is predefined-time-stable, with T_c a predefined time.

Proof. Consider the candidate Lyapunov function $W(\boldsymbol{x})$. Its derivative along the trajectories of system (6) closed by (7) is

$$\dot{W} = \frac{\partial W}{\partial \boldsymbol{x}} \left[-\frac{1}{T_c} \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}} \right]^T - k\boldsymbol{x} + \boldsymbol{\Delta} \right]$$

$$= -\frac{1}{T_c} + \frac{\partial W}{\partial \boldsymbol{x}} \left[-k\boldsymbol{x} + \boldsymbol{\Delta} \right].$$

By the Cauchy-Schwarz inequality:

(i) $\frac{\partial W}{\partial \boldsymbol{x}} \boldsymbol{x} \leq \left| \frac{\partial W}{\partial \boldsymbol{x}} \boldsymbol{x} \right| \leq \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right| ||\boldsymbol{x}||;$

(ii) $\left| \frac{\partial W}{\partial \boldsymbol{x}} \boldsymbol{\Delta} \right| \leq \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right| ||\boldsymbol{\Delta}|| \leq \delta \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right| ||\boldsymbol{x}||.$

Replacing the above points into the time derivative $\dot{W}$ expression, it yields

$$\dot{W} \leq -\frac{1}{T_c} - \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right| (k - \delta) ||\boldsymbol{x}|| \leq -\frac{1}{T_c}.$$

By Theorem 10, the origin of system (6) closed by (7) is predefined-time-stable, with T_c a predefined time.

Theorem 26. Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a non-vanishing bounded perturbation such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq \delta$, with $0 < \delta < \infty$ a known constant, and assume there exists a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying *(i), (ii), (iii)* and *(iv)* of Lemma 14. Selecting the control input as

$$\boldsymbol{u} = -\frac{1}{T_c} \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}} \right]^T - k \lfloor \boldsymbol{x} \rceil^0, \qquad (8)$$

with $T_c > 0$ and $k \geq \delta$, the origin of system (6) closed by (8) is predefined-time-stable, with T_c a predefined time.

Proof. Similar to the proof of Theorem 25.

Remark 27. The controller (8) in Theorem 26 contains the discontinuous term $k \lfloor \boldsymbol{x} \rceil^0$ to cancel the effect of the non-vanishing perturbation term.

5.2 Perturbation rejection via integral terms

Theorem 28. Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a non-vanishing bounded perturbation such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq \delta$, with $0 < \delta < \infty$ a known constant, and assume there exists a function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying *(i), (ii), (iii)* and *(iv)* of Lemma 14. Then, splitting the control input $\boldsymbol{u}$ into two parts, $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$, and selecting

(i) $\boldsymbol{u}_0 = -\frac{1}{T_c} \left|\left| \frac{\partial W}{\partial \boldsymbol{x}} \right|\right|^{-2} \left[\frac{\partial W}{\partial \boldsymbol{x}} \right]^T$, with $T_c > 0$ and

(ii) $\boldsymbol{u}_1 = -k \lfloor \boldsymbol{s} \rceil^0$, with $k > \delta$, and the auxiliary sliding variable $\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{z}$, where $\boldsymbol{z}$ is an integral variable, solution of $\dot{\boldsymbol{z}} = -\boldsymbol{u}_0$, $\boldsymbol{z}(0) = -\boldsymbol{x}(0)$,

the origin of the closed-loop system (6) is predefined-time-stable, and $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Proof. By the definition of $\boldsymbol{s}$, $\boldsymbol{z}$ and $\boldsymbol{u}_1$, the dynamics of $\boldsymbol{s}$ is obtained as

$$\dot{\boldsymbol{s}} = \dot{\boldsymbol{x}} + \dot{\boldsymbol{z}}$$
$$= \boldsymbol{u}_0 + \boldsymbol{u}_1 + \boldsymbol{\Delta} + \dot{\boldsymbol{z}}$$
$$= \boldsymbol{u}_1 + \boldsymbol{\Delta}$$
$$= -k \lfloor \boldsymbol{s} \rceil^0 + \boldsymbol{\Delta}. \qquad (9)$$

Consider the Lyapunov function candidate $V(\boldsymbol{s}) = ||\boldsymbol{s}||$; its derivative along the trajectories of (9) is

$$\dot{V} = \lfloor \boldsymbol{s}^T \rceil^0 \left(-k \lfloor \boldsymbol{s} \rceil^0 + \boldsymbol{\Delta} \right) = -k + \delta < 0.$$

Then, the origin of (9) is finite-time stable, and since $\boldsymbol{z}(0) = -\boldsymbol{x}(0)$, then $\boldsymbol{s}(0) = \boldsymbol{0}$, thus $\boldsymbol{s}(t) = \boldsymbol{0}$ for all $t \geq 0$.

From $\dot{\boldsymbol{s}} = 0$, the following equivalent control value is obtained as Utkin (1992):

$$\boldsymbol{u}_{1,eq} = -\boldsymbol{\Delta}.$$

Therefore, the sliding mode dynamics of (6)

$$\dot{\boldsymbol{x}} = \boldsymbol{u}_0.$$

is invariant with respect to the perturbation term $\boldsymbol{\Delta}$. By the definition of $\boldsymbol{u}_0$ and direct application of Lemma 14, the desired result is obtained.

Remark 29. The information of initial condition $\boldsymbol{x}(0)$ can be provided to the controller since the signal $\boldsymbol{x}(t)$ is available for all $t \geq 0$.

Example 30. For the dynamical system (6), $\boldsymbol{x} \in \mathbb{R}^3$ is required to satisfy $\boldsymbol{x}(t) = \boldsymbol{x}_{ss} = [3 \quad 0 \quad -1]^T$ for $t > 0.1 = T_c$. For this purpose $\boldsymbol{u}$ is selected as in Theorem 28, with the particular selection of $W(\boldsymbol{x} - \boldsymbol{x}_{ss}) = \frac{2}{\pi} \tan^{-1}(||\boldsymbol{x} - \boldsymbol{x}_{ss}||^{\frac{1}{2}})$. The perturbation term is chosen as $\boldsymbol{\Delta}(t, \boldsymbol{x}) = \frac{\sin(t)}{\sqrt{3}} [1 \quad 1 \quad 1]^T$, such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq 1 = k$. The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4}.
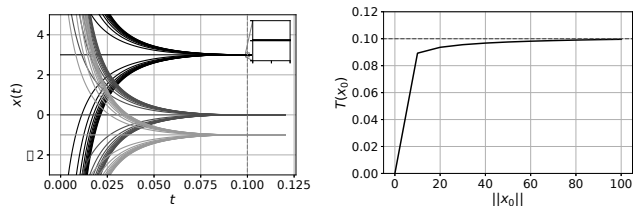
Fig. 1: Evolution of the states. Fig. 2: Settling-time function.

Fig. 1 shows the trajectories of the system for several initial conditions. All these trajectories converge to the equilibrium point x_{ss} at least in the predefined time T_c.

Fig. 2 shows the settling-time function obtained from the simulations. It can be seen that $\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c$, as stated in Theorem 28.

6. CONCLUSION

The stability analysis and characterization of a class of predefined-time dynamical systems was addressed in this note in the framework of the Lyapunov theory. In addition, the presented results give a path to design predefined-time-stable system using nonlinear non-smooth functions. Furthermore, some well-established previous results were generalized getting rid of the exponential characterization of this type of dynamical systems. Thus, the main conclusion of this work is that the developed methodology allows to define a more general, and some times numerically more stable kind of nonlinear controllers for stabilizing dynamical systems in predefined-time.

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