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## Robust Nested Sliding Mode Integral Control for Anti-lock Brake System

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**Abstract:** An integral nested Sliding Mode (SM) Block Control is proposed to control an Anti-lock Brake System (ABS) by employing integral SM and nested SM concepts. The control problem is to achieve reference tracking for the slip rate, such that, the friction between tyre and road surface is good enough to control the car. The closed-loop system is robust in presence of matched and unmatched perturbations. To show the performance of the proposed control strategy, a simulation study is carried on, where results show good behaviour of the ABS under variations in the road friction.

**Keywords:** Anti-lock Brake System (ABS), Sliding Mode Control, Integral Control, Automotive Control

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## 1 Introduction

The ABS control problem consists of imposing a desired vehicle motion and as a consequence, provides adequate vehicle stability. The main difficulty arising in the ABS design is due to its high non-linearities and uncertainties presented in the mathematical model. Therefore, the ABS has become an attractive research area in non-linear systems control framework. On the other hand, sliding mode (SM) approaches have been widely used for the problems of dynamic systems control and observation due to their characteristics of finite time convergence, robustness to uncertainties and insensitivity to external bounded disturbances (Utkin et al. 2009), (DeCarlo et al. 2011). Then, SM control emerges as an very interesting alternative for ABS design.

Several researchers have dealt with the issue of designing SM controllers and observers for automotive applications (Imine et al. 2011). For the ABS case, some

examples are Tan & Chin (1991), Unsal & Pushkin (1999), Hadri et al. (2002), Ming-Chin & Ming-Chang (2003), Patel et al. (2007), including the problem of extremum seeking presented in Drakunov et al. (1995).

Note that the SM techniques are based on the idea of the sliding manifold, that is an integral manifold for the closed-loop system with finite reaching time Drakunov & Utkin (1992). This manifold can be implemented by different methods including the use of a discontinuous function or continuous one with discontinuous derivatives (so-called Higher Order Sliding Modes). Let us note, that this issue of implementation, as demonstrated clearly in Utkin (1992) and earlier works is computational and depends on the system behaviour in the boundary layer of the sliding manifold. Thus, the main difficulty and innovations in continuous-time sliding mode research is in the design of the manifold rather than in the reaching phase that belongs more to numerical issue.

In this context, depending on the choice of the sliding manifold and its relative degree with respect to the control input, it is possible to find the so-called *unmatched perturbations*. Comparing these perturbations with the *matched ones*, it can be noted that the matched perturbations can be rejected directly by the control input while the unmatched ones affects the sliding mode equation and, as result, the closed-loop system behaviour (Drazenovich 1969).

Several methods has been treated in order to design a sliding manifold which is robust with respect to the unmatched perturbations. The discussions given in Estrada & Fridman (2010a) and Estrada & Fridman (2010b) present the use of Higher Order Sliding Modes (Levant 1998) for the finite time rejection of unmatched perturbations for a class of nonlinear systems presented in the Nonlinear Block Controllable form (Loukianov 1998).

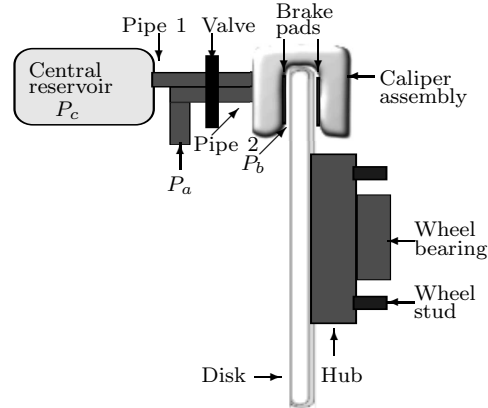
In this work we use an alternative approach, namely, the Integral Nested SM control (Huerta-Avila et al. 2008) which is based of Block Control technique (Drakunov et al. 1990a,b, Loukianov 1998, 2002) combined with the nested (Adhami-Mirhosseini & Yazdanpanah 2005) and integral (Utkin et al. 2009) sliding modes, providing robustness with respect to both the to matched and unmatched perturbations and ensuring output tracking. Moreover, it can be noted that, theoretically, the Integral Nested SM control can guarantee the robustness of the system throughout the entire response starting from the initial time instance and reduce the controller gains in comparison with the standard sliding mode (Rivera & Loukianov 2006, Huerta-Avila et al. 2007, González-Jiménez & Loukianov 2008). Our purpose in this work is to design an robust SM controller for the ABS which achieves asymptotic tracking the relative slip to a desired trajectory in presence of both the *matched* and *unmatched* external disturbances and parameter variations. The sliding manifold is designed on the basis of Integral Nested SM control due to its simplicity avoiding the formulation of exact SM differentiators (Levant 1998), which are needed in the two first mentioned SM methods for sliding manifold design. For the projection motion, we consider two situations. In the first one, it is considered the control input which can take only the values "0" and "1", that corresponds to the control of two position valves. It can be noted that this real situation was not considered in the previous works. A a first order SM controller is designed in order to ensure a good performance for the ABS. In the another situation, it is supposed that the control valve position is a continuous variable, and a Super-Twisting (ST) control

algorithm (Levant 1993, Fridman & Levant 2002) is implemented to achieve the designed sliding manifold be attractive. As a result, in both cases, the vehicle dynamics, i.e., the vehicle velocity, on the designed sliding manifold becomes asymptotically stable, ensuring asymptotic stability of the tracking error.

The work is organized as follows. The mathematical model for the longitudinal movement of a vehicle, including the brake system is presented in Section 2. In Section 3, an integral nested manifold for ABS is formulated and a first order SM and a Super-Twisting SM controllers, are designed. The simulation results are presented in Section 4 to verify the robustness and performance of the proposed control strategy. Finally, some conclusions are presented in Section 5.

## 2 Mathematical Model

In this section, the dynamic model of a vehicle is shown. Here we use a quarter of vehicle model, this model considers the pneumatic brake system, the wheel motion and the vehicle motion. We study the task of controlling the wheels rotation, such that, the longitudinal force due to the contact of the wheel with the road, is near from the maximum value in the period of time valid for the model. This effect is reached as a result of the ABS valve throttling.



**Figure 1:** Pneumatic brake model

### 2.1 Pneumatic Brake System Equations

The specific configuration of this system considers brake disks, which hold the wheels, as a result of the increment of the air pressure in the brake cylinder (Fig. 1). The entrance of the air through the pipes from the central reservoir and the expulsion from the brake cylinder to the atmosphere is regulated by a common valve. This valve allows only one pipe to be open, when 1 is open 2 is closed and vice versa. The time response of the valve is considered small, compared with the time constant of the pneumatic system.

Let us consider Fig. 1, we suppose the brake torque  $T_b$  is proportional to the pressure  $P_b$  in the brake cylinder

$$T_b = k_b P_b \quad (1)$$

with  $k_b > 0$ . For the brake system we use an approximated model of pressure changes in the brake cylinder due to the opening of the valve with a first order relation (Clover & Bernard 1998), this relationship can be represented as

$$\tau \dot{P}_b + P_b = P_c u \quad (2)$$

where  $\tau$  is the time constant of the pipelines,  $P_c$  is the pressure inside the central reservoir,  $u$  is the valve input signal.

We suppose two cases

**Case 2.1:** When the control valve position is a continuous variable, the parameter  $\tau$  of the equation (1) is constant.

**Case 2.2:**

When the control input can take only two values "0" or "1", the opening and closing of the valve is momentary and the parameter  $\tau$  of the equation (2) are given by the following rules:

- When pipe 1 is opened and 2 is closed then  $u = 1$  and  $\tau = \tau_{in}$
- When pipe 2 is opened and 1 is closed then  $u = 0$  and  $\tau = \tau_{out}$

For both cases, the atmospheric pressure  $P_a$  is considered equal to zero.

## 2.2 Wheel Motion Equations

To describe the wheels motion we will use a partial mathematical model of the dynamic system as is done in Novozhilov et al. (2000), Kruchinin et al. (2001), Petersen et al. (2001) and Magomedov et al. (2001).

Consider Figure 2, the dynamics of the angular momentum change relative to the rotation axis are given by

$$J\dot{\omega} = r f(s) - B_b \omega - T_b \quad (3)$$

where  $\omega$  is the wheel angular velocity,  $J$  is the wheel inertia moment,  $r$  is the wheel radius,  $B_b$  is a viscous friction coefficient due to wheel bearings and  $f(s)$  is the contact force of the wheel.

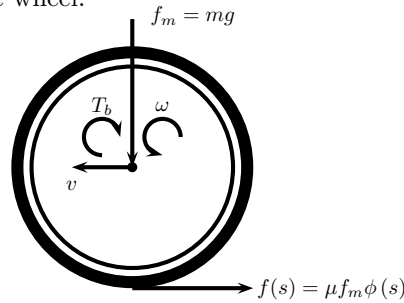


Figure 2: Wheel forces and torques

The expression for longitudinal component of the contact force in the motion plane is

$$f(s) = \mu f_m \phi(s) \quad (4)$$

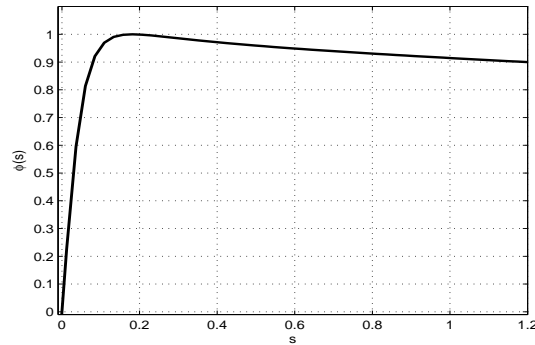
where  $\mu$  is the nominal friction coefficient between the wheel and the road,  $f_m$  is the normal reaction force in the wheel

$$f_m = mg + \Delta f_m(f_r, \dot{f}_r) \quad (5)$$

with  $m$  equal to the mass supported by the wheel,  $g$  is the gravity acceleration and  $\Delta f_m(f_r, \dot{f}_r)$  represents the variation of normal reaction force due to road perturbation,  $f_r$ , and its time derivative,  $\dot{f}_r$ . The function  $\phi(s)$  represents a friction/slip characteristic relation between the tire and road surface. Here, we use the Pacejka model (Bakker et al. 1989), defined as follows

$$\phi(s) = D \sin(C \arctan(Bs - E(Bs - \arctan(Bs))))).$$

In general, this model produces a good approximation of the tire/road friction interface. With the following parameters  $B = 10$ ,  $C = 1.9$ ,  $D = 1$  and  $E = 0.97$  that function represents the friction relation under a dry surface condition. A plot of this function is shown in Figure 3.



**Figure 3:** Characteristic function  $\phi(s)$

The slip rate  $s$  is defined as

$$s = \frac{v - r\omega}{v} \quad (6)$$

where  $v$  is the longitudinal velocity of the wheel mass centre. The equations (1)-(6) characterize the wheel motion.

### 2.3 The Vehicle Motion Equation

The vehicle longitudinal dynamics without lateral motion is considered. The main reasons for this assumptions are that the locked wheels generate forces on the car

which are in a direction opposite to the lineal wheel motion. Therefore, the steering angle changing has virtually no effect on the force vectors on the wheels. On the other hand, these forces in the lateral motion can be considered as perturbations for the longitude motion and can be rejected by the proposed controller.

Then, the vehicle longitudinal dynamics is written as

$$M\dot{v} = -F(s) - F_a(v) \quad (7)$$

where  $M$  is the vehicle mass,  $F_a(v)$  is the aerodynamic drag force, which is proportional to the vehicle velocity and is defined as

$$F_a(v) = \frac{1}{2}\rho C_d A_f (v + v_w)^2 + \Delta v_w$$

where  $\rho$  is the air density,  $C_d$  is the aerodynamic coefficient,  $A_f$  is the frontal area of vehicle,  $v_w$  is the wind velocity and  $\Delta v_w$  represents its variations.

As in the expression for longitudinal component of the contact force in the motion plane (4), the contact force of the vehicle  $F(s)$  is modelled of the form

$$F(s) = \mu\phi(s) f_M \quad (8)$$

where  $\mu$  is the nominal friction coefficient between the wheel and the road,  $f_M$  is the normal reaction force of the vehicle

$$f_M = Mg + \Delta f_M(f_r, \dot{f}_r) \quad (9)$$

with  $M$  equal to the vehicle mass,  $g$  is the gravity acceleration and  $\Delta f_M(f_r, \dot{f}_r)$  represents the variation of normal reaction force due to road perturbation,  $f_r$ , and its time derivative,  $\dot{f}_r$ .

The dynamic equations of the whole system (1)-(7) can be rewritten using the state variables

$$\mathbf{x} = [x_1, x_2, x_3]^T = [\omega, P_b, v]^T$$

with initial conditions  $x_0 = x(0)$  results the following form:

$$\begin{aligned} \dot{x}_1 &= -a_0 x_1 + a_1 f(s) - a_2 x_2 + \bar{\Delta}_1 \\ \dot{x}_2 &= -a_3 x_2 + bu + \bar{\Delta}_2 \\ \dot{x}_3 &= -a_4 F(s) - f_w(x_3) + \bar{\Delta}_3 \end{aligned} \quad (10)$$

with the output

$$y = s = h(x) = 1 - r \frac{x_1}{x_3}$$

where  $a_0 = B/J$ ,  $a_1 = r/J$ ,  $a_2 = k_b/J$ ,  $a_3 = 1/\tau$ ,  $a_4 = 1/M$ ,  $b = P_c/\tau$  and  $f_w(x_3) = \frac{1}{2M} (\rho C_d A_f) (x_3 + v_w)^2$ .

The term  $\bar{\Delta}_1$  contains the variations of the friction parameters  $\mu$ ,  $B_b$ , wheel inertia moment  $J$  and the normal reaction force due to road perturbation  $\Delta f_m(f_r, \dot{f}_r)$ . The term  $\bar{\Delta}_2$  contains the variations of the parameters  $\tau$  and  $P_c$ . Finally, the term  $\bar{\Delta}_3$  contains the variations of the parameters  $\mu$ ,  $C_d$ ,  $A_f$ ,  $\rho$ , the wind velocity variation  $\Delta v_w$  and the force due to road perturbation  $\Delta f_M(f_r, \dot{f}_r)$ .

### 3 Integral Nested Sliding Mode Control for ABS

Given  $s^*$  as the desired value of the relative slip  $s$ , which must be close to maximize the function  $\phi(s)$ , the considered problem is to design a controller that obtains reference tracking in despite of the perturbations in the system. As a solution, we propose an Integral Nested Sliding Mode controller (Huerta-Avila et al. 2007, 2008) for system (10).

Throughout the development of the controller, we will assume that all the state variables are available for measurement.

#### 3.1 Integral Sliding Manifold Design

Let  $s^*$  the slip reference, we define the output tracking error as

$$e_1 \triangleq x_1 - \frac{1 - s^*}{r} x_3. \quad (11)$$

Then, from (10) and (11) the derivative of  $e_1$  is

$$\dot{e}_1 = f_1(x_1, x_3) + b_1(x_1, x_3)x_2 + \Delta_1 \quad (12)$$

where  $f_1(x_1, x_3) = \frac{1-s^*}{r} [a_4 F(s) - f_w(x_3)] - a_0 x_1 + a_1 f(s)$  and  $b_1(x_1, x_3) = -a_2$ .

The term  $\Delta_1 = \bar{\Delta}_1 - \frac{1-s^*}{r} \bar{\Delta}_3$  will be considered as an unmatched and bounded perturbation term.

Considering the variable  $x_2$  as a virtual control in (12) we determinate its desired value  $x_{2\delta}$  as

$$x_{2\delta} = x_{2\delta,0} + x_{2\delta,1} \quad (13)$$

where  $x_{2\delta,0}$  is the nominal part of the virtual control and  $x_{2\delta,1}$  will be designed using the SM technique to reject the perturbation in (12) (Utkin et al. 2009).

In this way, we propose the desired dynamics for  $e_1$  as  $-k_0 e_0 - k_1 e_1$ , which is introduced by means of

$$x_{2\delta,0} = -\frac{1}{b_1(x_1, x_3)} [f_1(\mathbf{x}) + k_0 e_0 + k_1 e_1] \quad (14)$$

where  $k_0 > 0$ ,  $k_1 > 0$  and the new variable  $e_0$  is defined by

$$\dot{e}_0 = e_1, \quad e_0(0) = 0. \quad (15)$$

Now, in order to attenuate the perturbation term  $\Delta_1$  in (12), we define the pseudo sliding variable  $\sigma_1$  as

$$\sigma_1 = e_1 + z \quad (16)$$

where dynamics for the integral variable  $z$  will be defined later.

From (12), (13), (14) and (16) the derivative of  $\sigma_1$  is given by

$$\dot{\sigma}_1 = -k_0 e_0 - k_1 e_1 + x_{2\delta,1} + \Delta_1 + \dot{z}. \quad (17)$$



Selecting

$$\dot{z} = k_0 e_0 + k_1 e_1$$

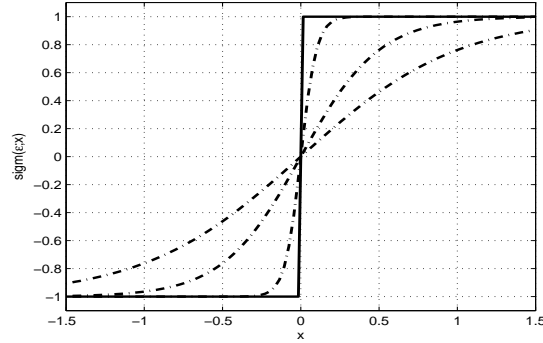
with  $z(0) = -e_1(0)$ , the equation (17) reduces to

$$\dot{\sigma}_1 = x_{2\delta,1} + \Delta_1. \quad (18)$$

To enforce pseudo sliding motion in (18) the virtual control  $x_{2\delta,1}$  is chosen as

$$x_{2\delta,1} = -k_{\sigma_1} \text{sigm}(\varepsilon, \sigma_1)$$

where we use the sigmoid function as a differentiable approximation to the sign function with the slope  $\varepsilon$ . Figure 4 shows the approximation for various values of the sigmoid function slope.



**Figure 4:** Sigmoid function for various values of the parameter  $\varepsilon$

Now, we define a new error variable  $e_2$  as

$$e_2 = x_{2\delta} - x_2. \quad (19)$$

Using (13) and (19), straightforward calculations reveal

$$\dot{e}_2 = -a_3 e_2 - bu + \Delta_2 \quad (20)$$

where the term

$$\Delta_2 = a_3 x_{2\delta} + \frac{\partial x_{2\delta}}{\partial x_1} \dot{x}_1 + \frac{\partial x_{2\delta}}{\partial x_3} \dot{x}_3 - \bar{\Delta}_2 \quad (21)$$

is considered as a perturbation.

Using the new variables  $e_0$ ,  $e_1$ ,  $e_2$  and  $\sigma_1$  the extended closed-loop system (12), (15), (20) and (18) is presented as

$$\dot{e}_0 = e_1 \quad (22)$$

$$\dot{e}_1 = -k_0 e_0 - k_1 e_1 + e_2 - k_{\sigma_1} \text{sigm}(\varepsilon, \sigma_1) + \Delta_1 \quad (23)$$

$$\dot{\sigma}_1 = -k_{\sigma_1} \text{sigm}(\varepsilon, \sigma_1) + \Delta_1 \quad (24)$$

$$\dot{e}_2 = -a_3 e_2 - bu + \Delta_2 \quad (25)$$

$$\dot{x}_3 = -a_4 F(s) - f_3(x_3) + \bar{\Delta}_3 \quad (26)$$

### 3.2 Sliding Mode Control for Two Position Valves

Considering the case 2.2, to induce sliding mode on the manifold  $e_2 = 0$  we choose the control  $u$  as

$$u = \frac{1}{2} \text{sign}(e_2) + \frac{1}{2}. \quad (27)$$

Now, the stability of (22) - (25) closed-loop by (27) is outlined in a step by step procedure:

**Step A)** Reaching phase of the projection motion (25);

**Step B)** SM stability of the projection motion (24);

**Step C)** SM stability of (22)-(23) in the vicinity of the manifold  $e_2 = 0$  and  $\sigma_1 = 0$ .

We use the following assumptions:

$$|\Delta_1| \leq \alpha_1 |\sigma_1| + \beta_1, \quad (28)$$

$$|\Delta_2| \leq \alpha_2 |e_2| + \beta_2 \quad (29)$$

and

$$\left| \dot{\Delta}_1 \right| \leq \alpha_0 |\dot{\sigma}_1| \quad (30)$$

with  $\alpha_0 > 0$ ,  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$ ,  $a_3 > \alpha_2$ , and  $b > |\Delta_2|$ .

**Step A)** The system (25) can be presented as follows:

CASE 1,  $e_2 < 0$ , then  $u = 0$  and

$$\dot{e}_2 = -a_3 e_2 + \Delta_2.$$

CASE 2,  $e_2 > 0$ , then  $u = 1$  and

$$\dot{e}_2 = -a_3 e_2 + \Delta_2 - b.$$

To analyse the stability conditions we use the Lyapunov function candidate  $V_2 = \frac{1}{2} e_2^2$ .

CASE 1 . The derivative of  $V_2$  with respect to time in this case is calculated as

$$\dot{V}_2 = e_2 (-a_3 e_2 + \Delta_2).$$

Under the condition (29), we have

$$\dot{V}_2 \leq -|e_2| ((a_3 - \alpha_2) |e_2| + \beta_2).$$

In this case, a solution of (25) with  $u = 0$  converges in a finite time to the region bounded by (Khalil 2001)

$$|e_2(t)| \leq \delta_0, \quad \delta_0 = \frac{\beta_2}{a_3 - \alpha_2}. \quad (31)$$

CASE 2 . Under the condition (29), we have

$$\dot{V}_2 \leq -|e_2|((a_3 - \alpha_2)|e_2| + (b - \beta_2)).$$

In this case, the solution  $e_2(t)$  of the subsystem (25) with  $u = 1$  converges in a finite time to zero ensuring sliding mode motion on  $e_2 = 0$ .

**Step B)** To analyse the stability of the subsystem (24) motion we assume that the  $\text{sign}(x)$  function can be approximated by the sigmoid function  $\text{sigm}(\varepsilon; x)$  in the form of the following equality:

$$\text{sign}(x) - \text{sigm}(\varepsilon; x) = \Delta_s(\varepsilon; x).$$

It is evidently that  $\Delta_s(x)$  is bounded, that is, for a given  $\varepsilon$  there is a positive constant  $0 < \gamma < 1$  such that

$$\|\Delta_s(\varepsilon; x)\| = \gamma$$

Using the Lyapunov function candidate

$$V_1 = \frac{1}{2}\sigma_1^2$$

and taking its derivative along the trajectories of (24) yields

$$\begin{aligned} \dot{V}_1 &= \sigma_1 [-k_{\sigma_1} \text{sigm}(\varepsilon, \sigma_1) + \Delta_1] \\ &\leq -|\sigma_1| [k_{\sigma_1} (1 - \gamma) - \alpha_1 |\sigma_1| - \beta_1] \end{aligned}$$

Therefore, under the condition

$$k_{\sigma_1} > \frac{\beta_1}{1 - \gamma}$$

in the region

$$|\sigma_1| < \frac{k_{\sigma_1} (1 - \gamma) - \beta_1}{\alpha_1}$$

$\sigma_1$  converges into a vicinity defined by

$$|\sigma_1| \leq \vartheta, \quad \vartheta = \frac{\ln\left(\frac{2-\gamma}{\gamma}\right)}{2\varepsilon}$$

ensuring in this vicinity

$$\dot{\sigma}_1 = \{-k_{\sigma_1} \text{sigm}(\varepsilon, \sigma_1)\}_{eq} + \Delta_1 = 0 \quad (32)$$

where  $\{\}_{eq}$  denotes an *equivalent value operator* of a function in sliding mode (Utkin 1992).

**Step C)** The sliding mode motion in the vicinity of the manifold  $e_2 = 0$  and  $\sigma_1 = 0$  is described the subsystem (22)-(23) reduced by using (32) to

$$\dot{e}_{01} = A_{01}e_{01} + b_{01}e_2 \quad (33)$$

where  $e_{01} = (e_0, e_1)^T$ ,  $A_{01} = \begin{bmatrix} 0 & 1 \\ -k_0 & -k_1 \end{bmatrix}$  and  $b_{01} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

Using a positive definite solution  $P_{01}$  of the Lyapunov equation  $P_{01}A_{01} + A_{01}^T P_{01} = I_2$ , it is easy to show that the solutions of the SM perturbed equation (33) under the inequality (31) is ultimately bounded by (Khalil 2001)

$$\|e_{01}(t)\| \leq \delta_{01}, \quad \delta_{01} = \sqrt{\frac{\lambda_{\max}(P_{01})}{\lambda_{\min}(P_{01})} \frac{\delta_0}{\theta}}, \quad 0 < \theta < 1.$$

### 3.3 Sliding Mode Control for Continuous Position Valves

We now consider the types of valve that can vary its position in a continuous range. To induce sliding mode in the subsystem (25) on the manifold  $e_2 = 0$ , the super-twisting control algorithm is applied (Levant 1993, Fridman & Levant 2002)

$$\begin{aligned} u &= \frac{\lambda_1}{b} |e_2|^{\frac{1}{2}} \text{sign}(e_2) - u_1 \\ \dot{u}_1 &= -\lambda_2 \text{sign}(e_2). \end{aligned} \quad (34)$$

Equation (25) closed by the control (34) results in

$$\begin{aligned} \dot{e}_2 &= -\lambda_1 |e_2|^{\frac{1}{2}} \text{sign}(e_2) + u_1 + \psi_2 \\ \dot{u}_1 &= -\lambda_2 \text{sign}(e_2) \end{aligned} \quad (35)$$

where  $\psi_2 = -a_3 e_2 + \Delta_2$ . By using (20) and (21) one can write

$$\psi_2 = a_3 x_2 + \frac{\partial x_{2\delta}}{\partial x_1} \dot{x}_1 + \frac{\partial x_{2\delta}}{\partial x_3} \dot{x}_3 \leq \bar{\beta}_2.$$

To analyse stability conditions, the following candidate Lyapunov function (Moreno & Osorio 2008) is used:

$$\begin{aligned} \mathcal{V} &= 2\lambda_2 |e_2| + \frac{1}{2} u_1^2 + \frac{1}{2} (\lambda_1 |e_2|^{1/2} \text{sign}(e_2) - u_1)^2 \\ &= \xi^T P \xi \end{aligned}$$

where  $\xi^T = (|e_2|^{1/2} \text{sign}(e_2), u_1)$  and  $P = \frac{1}{2} \begin{pmatrix} 4\lambda_2 + \lambda_1^2 - \lambda_1 & \\ -\lambda_1 & 2 \end{pmatrix}$ .

Calculating its time derivative along the solution of (35) yields

$$\dot{\mathcal{V}} = -\frac{1}{|e_2|^{1/2}} \xi^T Q \xi + \frac{\psi_2}{|e_2|^{1/2}} q_1^T \xi$$

where

$$Q = \frac{\lambda_1}{2} \begin{pmatrix} 2\lambda_2 + \lambda_1^2 - \lambda_1 & \\ -\lambda_1 & 1 \end{pmatrix}, \quad q_1^T = (2\lambda_2 + \frac{1}{2}\lambda_1^2, -\frac{1}{2}\lambda_1).$$

Moreover, one can easily see that

$$\frac{\psi_2}{|e_2|^{1/2}} q_1^T \xi \leq \frac{\bar{\beta}_2}{|e_2|^{1/2}} \xi^T Q_1 \xi$$

with

$$Q_1 = \begin{pmatrix} 2\lambda_2 + \frac{1}{2}\lambda_1^2 & 0 \\ 0 & -\frac{1}{2}\lambda_1 \end{pmatrix}.$$

Therefore, the derivative of the Lyapunov function is simplified to

$$\dot{V} = -\frac{k_1}{2|e_2|^{1/2}} \xi^T \tilde{Q} \xi$$

where

$$\tilde{Q} = \begin{pmatrix} \lambda_1 \lambda_2 + \frac{1}{2}\lambda_1^3 + (2\lambda_2 + \frac{1}{2}\lambda_1^2)\bar{\beta}_2 & -\lambda_1 \\ -\lambda_1 & 1 - \frac{1}{2}\lambda_1 \bar{\beta}_2 \end{pmatrix}$$

In this case the controller gains  $\lambda_1$  and  $\lambda_2$  can easily be chosen such that  $\tilde{Q} > 0$ , implying that the derivative of the Lyapunov function is negative definite. Finally the analysis can be continued as in **Step B** of the above subsection.

#### 4 Simulation Results

To show the effectiveness of the proposed control law, simulations have been carried out on the wheel model design example, the system parameters used are listed in Table 1.

Parameter	Value	Parameter	Value
$A_f$	6.6	$V_w$	-6
$P_c$	8	$v$	0.5
$M$	1800	$B$	10
$J$	18.9	$C$	1.9
$r$	0.35	$D$	1
$m$	450	$E$	0.97
$\rho$	1.225	$g$	9.81
$C_d$	0.65	$P_a$	0

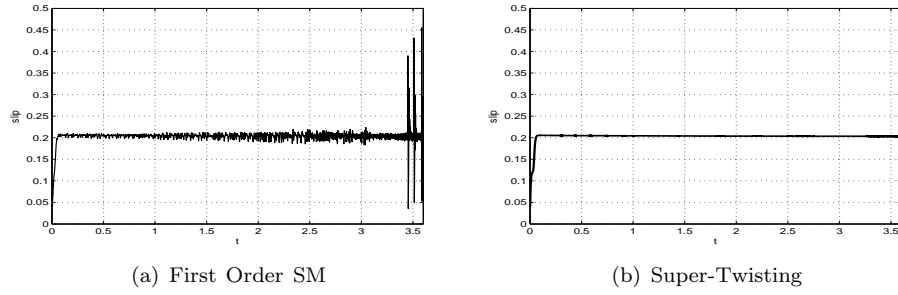
In order to maximize the friction force, we suppose that slip tracks a constant signal during the simulations

$$s^* = 0.203$$

which produces a value close to the maximum of the function  $\phi(s)$ . The parameters used in the control law are  $k_0 = 700$ ,  $k_1 = 120$ ,  $k_3 = 2$ ,  $k_4 = 100$ ,  $k_{\sigma_1} = 10$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\varepsilon = 100$ .

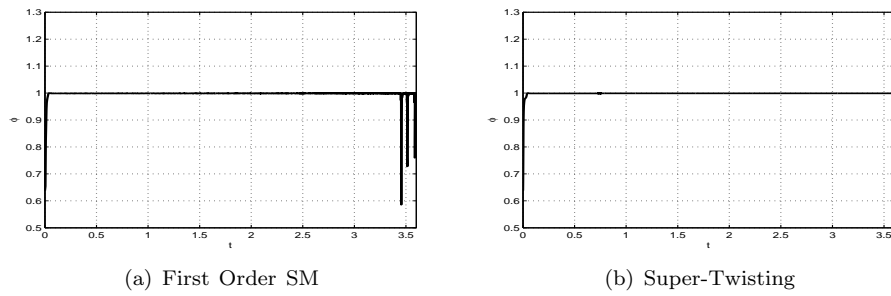
On the other hand, to show robustness property of the control algorithm in presence of parametric variations we introduce a change of the friction coefficient  $\mu$  which produces different contact forces, namely  $F$  and  $\hat{F}$ . Then,  $\mu = 0.5$  for  $t < 1$  s,  $\mu = 0.52$  for  $t \in [1, 2.5)$  s, and  $\mu = 0.5$  for  $t \geq 2.5$  s. It is worth mentioning that just the nominal values were considered in the control design.

In Figures 5a and 5b the slip  $s$  performance trough the simulation is shown



**Figure 5:** Slip performance in the braking process

Figures 6a and 6b shows the friction function behaviour  $\phi(s)$  during the braking process



**Figure 6:** Performance of  $\phi(s)$  in the braking process

while Figures 7a and 7b summarize the behaviour of the error variable  $e_1$ .

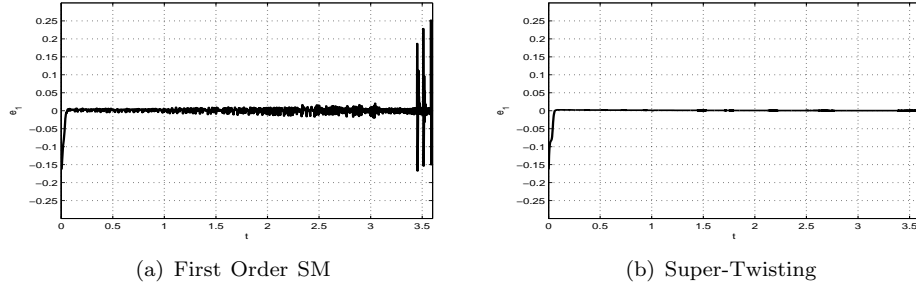


Figure 7: Tracking error  $e_1 = s - s^*$

and Figures 8a and 8b shows the error on the sliding manifold

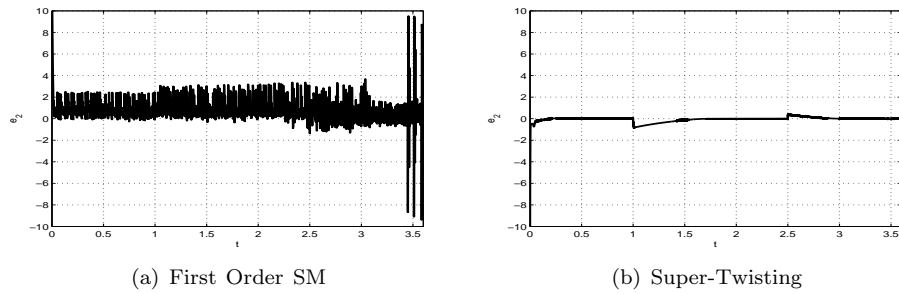


Figure 8: Sliding manifold error

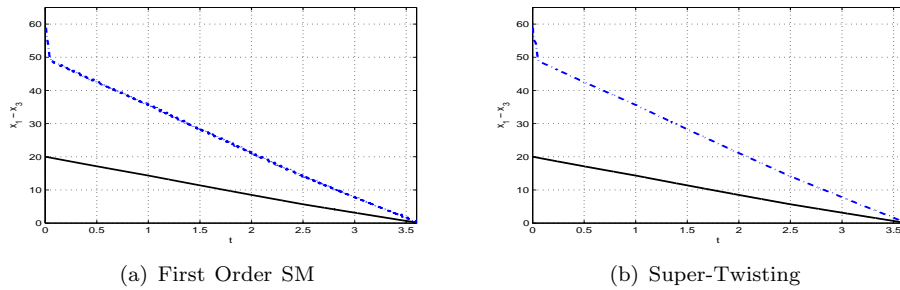
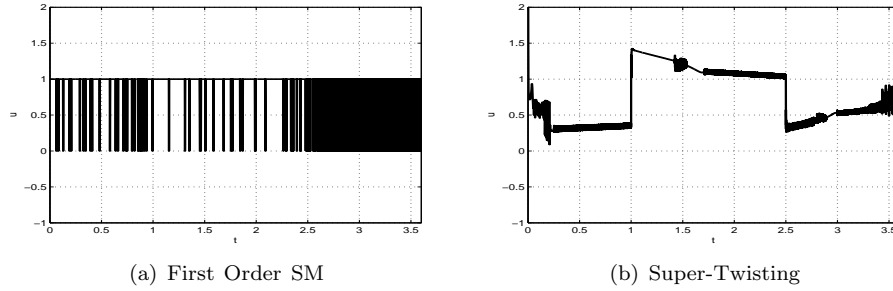


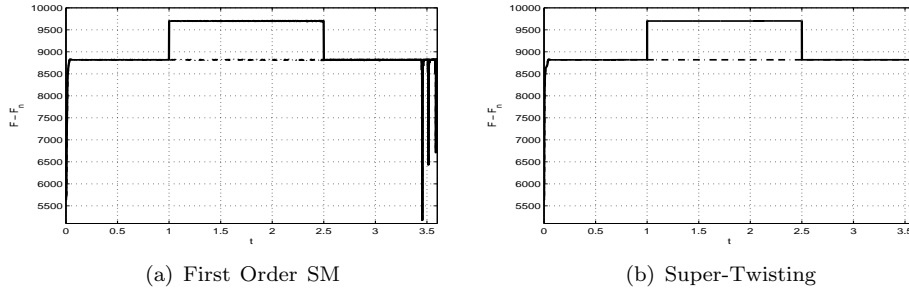
Figure 9: Longitudinal speed  $v$  (solid) and linear wheel speed  $r\omega$  (dashed)

In Figures 9a and 9b the longitudinal speed  $v$  and the linear wheel speed  $r\omega$  are shown; it is worth noting that the slip controller should be turn off when the longitudinal speed  $v$  is close to zero. Figures 10a and 10b the control action is shown.



**Figure 10:** Control input  $u$

Finally, in Figures 11a and 11b the nominal  $F$ , and the  $\hat{F}$  contact force are shown.



**Figure 11:** Nominal  $F$  (dashed) and perturbed  $\hat{F}$  (solid) vehicle contact forces

## 5 Conclusion

In this work an Integral Nested SM controller for ABS has been proposed for the cases of discontinuous and continuous valve action. The simulation results show good performance and robustness of the designed closed-loop system in presence of both, the matched and unmatched perturbations, included parametric variations and unmodelled dynamics, giving an important application of the SM control theory in the automotive problems. Therefore, the ABS can cope very well with the SM control which can be applied in a straight fashion in the both cases: continuous and discontinuous actuators, showing in that way a clear advantage



over another control techniques, where the presence of discontinuous elements can not be treated in a natural way.

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