GPS BASED DESIGN OF THE LOCAL CLOCK CONTROL SYSTEM
BASED ON THE OPTIMALLY UNBIASED MOVING AVERAGE FILTER

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Abstract

In this paper we made the simulation steering of the local clock time errors with simple
moving average (MA), optimally unbiased moving average (OMA), the two and three-state
Kalman filters. The references signal (precise time) was suministred by GPS. In this task we
have two important activities, estimating and the error control, so the principal parameter in
this study is the root mean square error (RMSE) of steering. When steering the GPS-based
time error in the local clock with four filters, we found out that, of the filter with the same
time constant, the optimally unbiased MA filter desmostred the steering error between the two
and three state Kalman filter.

INTRODUCTION

To be outside of synchronization is be outside of control, without precise time
reference in applications that requiered of synchronization, so we are undefeat in problems
related with the time. It is well known that timekeping solve two important tasks: estimating
the time error of a local clock with reference time signal (GPS) and elimination with the
synchronization loop. In our particular case we take the synchronization loop as an optimal
stochastical error control. Optimal lineal stochastic estimation theory, named Kalman filtering
theory [1], solves both tasks straightforwardly. In this task we have three important variables
for simulation, N is the points numbers in the average for FIR filters (simple MA and OMA
filter), m is the points numbers taken for the lineal regression (LR), so LR is used for estimate
the first values of time error, frequency and frequency aging of the local clock, and K3 is the
control coefficient for to obtain the least RMSE of steering.

To determine the tradeoff, we examine each filter for time error model identified by
the finite polynomial [2]

\[ x_n = x_0 + y_0 \Delta n + \frac{D}{2} \Delta^2 n^2 + w_n, \]  

(1)

where \( n = 0, 1, \ldots \); \( \Delta = t_n - t_{n-1} \) is the sample time; \( t_n \) is the discrete time; \( x_0 \) is the initial time
error; \( y_0 \) is the initial fractional frequency offset of a local clock from the reference
frequency; \( D \) is the lineal fractional frequency drift rate; and \( w_n(t) \) is the random time error
deveiation component. For the simulation steering we have an important formula for determine
the time constant

\[ T_r = \frac{T(N-1)}{3600}, \]  

(2)

where \( T \) is the digitalization time in seconds; 3600 equal in seconds per hour.
DEVELOPMENT

The signal for simulation steering is showed in figure 1, so it has the reference signal with respect of the local clock, the observation measured with respect to GPS timing and additive Gaussian noise with mean zero.

As a first step, we analyze the simple MA filter, so this filter produces the lowest noise for stationary processes, but in non-stationary processes is not good, the estimate equation of time error for simple MA filter is

$$\hat{e}_n = \frac{1}{N} \sum_{i=0}^{N-1} e_{n-i},$$  \hspace{1cm} (3)

where $e_{n-i}$ is the observation signal.

In (3) the constant coefficient $\frac{1}{N}$ produce inherently the bias.

For the steering task, the simple MA algorithm is give it in controlled estimated (4) and controlled observation (5)

$$\lambda ma_n = \frac{1}{N} \sum_{i=0}^{N-1} \xi ma_{n-i},$$  \hspace{1cm} (4)

$$\xi ma_{n+1} = \xi G_{n+1} - K3\lambda ma_n.$$  \hspace{1cm} (5)

The optimally unbiased MA filter for estimate the time error is defined by

$$\xi_n = \sum_{i=0}^{N-1} \frac{2(2N-1)-6i}{N(N+1)} e_{n-i},$$  \hspace{1cm} (6)

in (7) we have the weightig function as

$$W_i(N) = \begin{cases} 
2(2N-1)-6i & 0 \leq i \leq N-1 \\
N(N+1) & otherwise 
\end{cases}$$  \hspace{1cm} (7)
this evidently is nonzero on the averaging interval only [3]. In case of steering algorithm with optimally unbiased MA filter, we have the next equations

\[
\xi \text{om}a_{n1} = \frac{N}{N+1} \xi \text{om}a_{n1-1} + \xi \text{om}a_n \quad (8)
\]

\[
\xi \text{om}a_{n+1} = \xi G_{n1} - K3\xi \text{om}a_n \quad (9)
\]

For two and three state Kalman filter, we have the next equations, in (10) it's the general form for observation signal, where \( H_n \) is the measurement matrix, \( u_n \) is a control vector, \( n_n \) is a Gaussian noise vector

\[
\xi_n = H_n \lambda_n + u_n + n_n \quad (10)
\]

\[
\lambda_n = A_{n-1} \lambda_n + n \quad (11)
\]

\[
H_n = [1 0], H_n = [1 0 0] \quad (12)
\]

In (13) we can see the vector states of local clock like time error \( x_n \), frequency offset \( y_n \), and frequency aging \( \alpha_n \).

\[
\lambda_n = \begin{bmatrix} x_n \\ y_n \\ \alpha_n \end{bmatrix}, \quad \lambda_n = \begin{bmatrix} x_n \\ \alpha_n \end{bmatrix} \quad (13)
\]

So, in (14) Kalman algorithm required a transition matrix related the sample time and in (15) are the covariance matrix

\[
A_n = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}, \quad A_n = \begin{bmatrix} 1 & \Delta & \Delta^2/2 \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix} \quad (14)
\]

\[
\Psi_{x_n} = \frac{N_{nn}}{2} \begin{bmatrix} \Delta^2/3 & \Delta/2 & \Delta/2 \\ \Delta/2 & 1 & \Delta/2 \\ \Delta/2 & \Delta/2 & 1 \end{bmatrix}, \quad \Psi_n = E\{n_n n_n^T\} = \begin{bmatrix} \Psi_{11n} & \Psi_{12n} & \Psi_{13n} \\ \Psi_{21n} & \Psi_{22n} & \Psi_{23n} \\ \Psi_{31n} & \Psi_{32n} & \Psi_{33n} \end{bmatrix} \quad (15)
\]

After of this, the algorithm for estimated (16) and steering (17)-(18) by means of the Kalman filter are:

\[
\hat{\lambda}_n = A_{n-1} \hat{\lambda}_{n-1} + K_n \left( \xi_n - u_n - H_n A_{n-1} \hat{\lambda}_{n-1} \right) \quad (16)
\]

\[
\lambda e_n = A_{n-1} \lambda e_{n-1} + K_n \left( \xi_n - H_n A_{n-1} \lambda e_{n-1} \right) \quad (17)
\]

\[
\xi_{n+1} = \xi G_{n1} - K3\lambda e_n \quad (18)
\]

where \( K_n \) is the Kalman gain, \( \xi G_{n1} \) is the observation signal from PRN file.

We determined the statistical steering error by means a general error function, this general error function is similar for the four filters, so its structure was determined by the time interval \( M = 0 \cdots m - N - 1 \).

\[
\lambda \text{om}a_{M+1} = \lambda_{M+1} - K3\lambda \text{om}a_{M+1} \quad (19)
\]
where \( n_i \) is the length of the observation file, in the right-part of (19) we show only the time error controlled for optimally unbiased MA filter, but we can replace easily it for any time error controlled like simple MA, two or three state Kalman filter.

Now, the bias \( \Delta \hat{\tau} = E[\lambda f_c] \), the variance \( \sigma^2 = E[(\lambda f_c - \Delta \hat{\tau})^2] \), root mean square deviation (RMSD) \( \sigma = \sigma^2 \), root mean square error (RMSE) \( \varepsilon_{\text{RMS}} = E[\lambda f_c^2] = \Delta \hat{\tau}^2 + \sigma^2 \) and maximal error \( \varepsilon_{\text{global}} = 0.5(\varepsilon_{\text{RMS}} + \lambda f_{c_{\text{max}}}) \) of steering were calculated from (19).

**Simulation and Conclusions**

As the principal goal was the steering of the time error in the local clock, in figure 2 we have all steering error for \( N \) average points (order filter), the simulation was evaluated in Matlab and Mathcad Software. As we can see, the worst filter was the simple MA and this was because of the process is non-stationary, the best filter is the two state Kalman filter and the optimally unbiased MA filter is situated between two and three state Kalman filter; when filter order increases, transition time increases too, therefore a filter with \( N \) close to 80 or 100 average points is recommended for steering error of the local clock. The FIR filter did not require any a priori knowledge about the GPS-based time error process in the opposite with Kalman filters [4].

![Figure 2 – Least RMSE of steering with all filters.](image)

**References**