On Optimal Predefined-Time Stabilization

Esteban Jiménez-Rodríguez[†] **Juan Diego Sánchez-Torres**[‡]
Alexander G. Loukianov[†]

[†] *Department of Electrical Engineering, CINVESTAV-IPN, Av. del Bosque 1145 Col. El Bajío C.P. 45019, Guadalajara, Jalisco, México. E-mail: {ejimenezr,louk}@gdl.cinvestav.mx*
[‡] *Department of Mathematics and Physics, ITESO, Periférico Sur Manuel Gómez Morín 8585 C.P. 45604, Tlaquepaque, Jalisco, México. E-mail: dsanchez@iteso.mx*

Abstract: This paper addresses the problem of optimal predefined-time stability. Predefined-time stable systems are a class of fixed-time stable dynamical systems for which the minimum bound of the settling-time function can be defined a priori as a explicit parameter of the system. Sufficient conditions for a controller to solve the optimal predefined-time stabilization problem for a given system are provided. These conditions involve a Lyapunov function that satisfy both a certain differential inequality for guaranteeing predefined-time stability and the steady-state Hamilton-Jacobi-Bellman equation for guaranteeing optimality. Finally, this result is applied to the predefined-time optimization of the sliding manifold reaching phase.

Keywords: Hamilton-Jacobi-Bellman Equation, Lyapunov Functions, Optimal Control, Predefined-Time Stability.

1. INTRODUCTION

Finite-time stable dynamical systems provide solutions to applications which require hard time response constraints. Important works involving the definition and application of finite-time stability have been carried out in Roxin (1966); Haimo (1986); Utkin (1992); Bhat and Bernstein (2000); Moulay and Perruquetti (2005, 2006). Nevertheless, this finite stabilization time is often an unbounded function of the initial conditions of the system. Making this function bounded to ensure the settling time is less than a certain quantity for any initial condition may be convenient, for instance, for optimization and state estimation tasks. With this purpose, a stronger form of stability, in which the convergence time presents a class of uniformity with respect to the initial conditions, called *fixed-time stability* was introduced. The notion of fixed-time stability is presented in Andrieu et al. (2008) for homogeneous systems and it was proposed in Cruz-Zavala et al. (2010); Polyakov (2012); Polyakov and Fridman (2014) for systems with sliding modes.

When fixed-time stable dynamical systems are applied to control or observation, it may be difficult to find a direct relationship between the gains of the system and the upper bound of the convergence time; thus, tuning the system in order to achieve a desired maximum stabilization time is not a trivial task. A simulation-based approximation to select the values of the tuning parameters is proposed in Fraguela et al. (2012) under the concept of *prescribed-time stability*; this method permits to design robust sliding differentiators for noisy signals by expressing the gains as functions of the desired settling time. Therefore, prescribed-time stable systems present a way to surmount the tuning problem. However, this prescribed time usually constitutes a conservative estimation of the upper bound of the convergence time; that is, the prescribed time is commonly larger, maybe quite larger, than the *true* amount of time the system takes to converge.

To overcome the above, another class of dynamical systems which exhibit the property of *predefined-time stability*, have been studied (Sánchez-Torres et al., 2014; Sánchez-Torres et al., 2015). For this systems the prescribed-time stability coincides with the fixed-time stability when the true settling time is considered. The upper bound for the convergence time of the proposed kind of systems appears explicitly in their dynamical equations; in particular, it equals the reciprocal of the system gain. This bound is not a conservative estimation but truly the minimum value that is greater than all the possible exact settling times. All the mentioned properties of predefined-time stable systems are characterized by a suitable Lyapunov theorem.

On the other hand, the infinite-horizon, nonlinear nonquadratic optimal asymptotic stabilization problem was addressed in Bernstein (1993). The main idea of the results are based on the condition that a Lyapunov function for the nonlinear system is at the same time the solution of the steady-state Hamilton-Jacobi-Bellman equation, guaranteeing both asymptotic stability and optimality. Nevertheless, returning to the first paragraph idea, the finite-time stability is a desired property in some applications, but optimal finite-time controllers obtained using the maximum principle do not generally yield feedback controllers. In this sense, the optimal finite-time stabilization is studied in Haddad and L'Afflitto (2016), as an extension of Bernstein (1993). Since the results are based on the framework developed in Bernstein (1993), the controllers obtained are in fact feedback controllers.

Consequently, as an extension of the ideas presented in Bernstein (1993); Sánchez-Torres et al. (2015); Haddad and L'Afflitto (2016), this paper addresses the problem of *optimal predefined-time stabilization*, namely the problem of finding a state-feedback control that minimizes certain performance measure, guaranteeing at the same time predefined-time stability of the closed-loop system. In particular, sufficient conditions for a controller to solve the optimal predefined-time stabilization problem for a given system are provided. These conditions involve a Lyapunov function that satisfy both a certain differential inequality for guaranteeing predefined-time stability and the steady-state Hamilton-Jacobi-Bellman equation for guaranteeing optimality. Finally, this result is applied to the predefined-time optimization of the sliding manifold reaching phase.

In the following, Section 2 presents the mathematical preliminaries needed to introduce the proposed results. Section 3 exposes the main results of this paper, which are the sufficient conditions for a controller to solve the optimal predefined-time stabilization problem and a particularization to affine systems. Section 4 shows the application of the obtained results to the predefined-time optimization of the sliding manifold reaching phase, and the simulation results are shown in Section 5. Finally, Section 6 presents the conclusions of this paper.

2. MATHEMATICAL PRELIMINARIES

2.1 Predefined-Time Stability

Consider the system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function such that $f(0) = 0$, i.e.the origin is an equilibrium point of (1).

First, the concepts of finite-time, fixed-time and predefined-time stability are reviewed.

Definition 2.1. (Polyakov, 2012) The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$.

Definition 2.2. (Polyakov, 2012) The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.

Remark 2.1. Note that there are several choices for $T_{\max}$. For instance, if the settling-time function is bounded by $T_{\rm m}$, it is also bounded by $\lambda T_{\rm m}$ for all $\lambda \geq 1$. This motivates the following definition.

Definition 2.3. (Sánchez-Torres et al., 2014) Let $\mathcal{T}$ be the set of all the bounds of the settling time function for the system (1), i.e.,

$$\mathcal{T} = \{T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}\}. \tag{2}$$

The minimum bound of the settling-time function T_f, is defined as:

$$T_f = \inf \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \tag{3}$$

Definition 2.4. (Sánchez-Torres et al., 2014) For the case of fixed time stability when the time T_f defined in (3) can be tuned by a particular selection of the parameters of the system (1), it is said that the origin of the system (1) is *predefined-time stable*.

The following Lyapunov-like lemma provides a characterization of predefined-time stability.

Lemma 2.1. (Sánchez-Torres et al., 2014) Assume there exist a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, and real numbers $\alpha > 0$ and $0 < p \leq 1$, such that:

$$V(0) = 0 \tag{4}$$

$$V(x) > 0, \qquad \forall x \neq 0, \tag{5}$$

and the derivative of V along the trajectories of the system (1) satisfies

$$\dot{V} \leq -\frac{\alpha}{p}\exp(V^p)V^{1-p}. \tag{6}$$

Then, the origin is globally predefined-time stable for (1) and $T_f = \frac{1}{\alpha}$.

2.2 Optimal Control

Consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{7}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system input, which is restricted to belong to a certain set $\mathcal{U} \subset \mathbb{R}^m$ of the admissible controls, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function with $f(0, 0) = 0$.

The system (7) is to be controlled to minimize the performance measure

$$J(x_0, u(\cdot)) = \int_0^{t_f} L(x(t), u(t))\mathrm{d}t, \tag{8}$$

where $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function, assumed to be convex in u. To this end, define the minimum cost function as

$$J^*(x(t), t) = \min_{u \in \mathcal{U}} \left\{ \int_t^{t_f} L(x(\tau), u(\tau))\mathrm{d}\tau \right\}. \tag{9}$$

Defining the *Hamiltonian*, for $p \in \mathbb{R}^n$ (usually called the costate)

$$\mathcal{H}(x, u, p) = L(x, u) + p^T f(x, u), \tag{10}$$

the *Hamilton-Jacobi-Bellman* (HJB) equation can be written as

$$0 = \min_{u \in \mathcal{U}} \left\{ \mathcal{H}\left(x, u, \frac{\partial J^*(x,t)}{\partial x}^T \right) \right\} + \frac{\partial J^*(x,t)}{\partial t}, \tag{11}$$

and it provides a sufficient condition for optimality.

For infinite-horizon problems (limit as $t_f \to \infty$), the cost does not depend on t anymore and the partial differential equation (11) reduces to the steady-state HJB equation

$$0 = \min_{u \in \mathcal{U}} \mathcal{H}\left(x, u, \frac{\partial J^*(x)}{\partial x}^T \right). \tag{12}$$

3. OPTIMAL PREDEFINED-TIME STABILIZATION

The main result of this paper is presented in this section. First, the notion of optimal predefined-time stabilization is defined.

Definition 3.1. Consider the optimal control problem for the system (7)

$$\min_{u\in\mathcal{U}(T_c)} J(x_0,u(\cdot)) = \int_0^\infty L(x(t),u(t))\mathrm{d}t \tag{13}$$

where

$$\mathcal{U}(T_c) = \{u(\cdot) : u(\cdot) \text{ stabilizes (7) in a predefined time } T_c\}.$$

This problem is called the *optimal predefined-time stabilization problem* for the system (7).

The following theorem gives sufficient conditions for a controller to solve this problem.

Theorem 3.1. Assume there exist a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, real numbers $T_c > 0$ and $0 < p \le 1$, and a control law $\phi^* : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$V(0) = 0 \tag{14}$$
$$V(x) > 0, \quad \forall x \neq 0, \tag{15}$$
$$\phi^*(0) = 0 \tag{16}$$
$$\frac{\partial V}{\partial x} f(x,\phi^*(x)) \le -\frac{1}{T_c p}\exp(V^p)V^{1-p} \tag{17}$$
$$\mathcal{H}\left(x,\phi^*(x),\frac{\partial V}{\partial x}^T\right) = 0 \tag{18}$$
$$\mathcal{H}\left(x,u,\frac{\partial V}{\partial x}^T\right) \ge 0, \quad \forall u \in \mathcal{U}(T_c). \tag{19}$$

Then, with the feedback control

$$u^*(\cdot) = \phi^*(x(\cdot)) = \arg\min_{u\in\mathcal{U}(T_c)} \mathcal{H}\left(x,u,\frac{\partial V}{\partial x}^T\right), \tag{20}$$

the origin of the closed-loop system

$$\dot{x}(t) = f(x(t),\phi^*(x(t))) \tag{21}$$

is predefined-time stable with $T_f = T_c$. Moreover, the feedback control law (20) minimizes $J(x_0,u(\cdot))$ in the sense that

$$J(x_0,\phi^*(x(\cdot))) = \min_{u\in\mathcal{U}(T_c)} J(x_0,u(\cdot)) \tag{22}$$
$$= V(x_0), \tag{23}$$

i.e., the feedback control law (20) solves the optimal predefined-time stabilization problem for the system (7).

Proof. Predefined-time stability with predefined time T_c follows directly from the conditions (14)-(17) and applying the Lemma 2.1 to the closed-loop system (21).

To prove (23), let $x(t)$ be the solution of system (21). Then,

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x} f(x(t),\phi^*(x(t))).$$

From the above and (18) it follows

$$\begin{aligned} L(x(t),\phi^*(x(t))) &= L(x(t),\phi^*(x(t))) + \\ &\quad \frac{\partial V}{\partial x} f(x(t),\phi^*(x(t))) - \dot{V}(x(t)) \\ &= \mathcal{H}\left(x(t),\phi^*(x(t)),\frac{\partial V}{\partial x}^T\right) - \dot{V}(x(t)) \\ &= -\dot{V}(x(t)). \end{aligned}$$

Hence,

$$\begin{aligned} J(x_0,\phi^*(x(\cdot))) &= \int_0^\infty -\dot{V}(x(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) \\ &= V(x_0). \end{aligned}$$

Now, to prove (22), let $u(\cdot) \in \mathcal{U}(T_c)$ and let $x(t)$ be the solution of (7), so that

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x} f(x(t),u(t)).$$

Then

$$\begin{aligned} L(x(t),u(t)) &= L(x(t),u(t)) + \frac{\partial V}{\partial x} f(x(t),u(t)) - \dot{V}(x(t)) \\ &= \mathcal{H}\left(x(t),u(t),\frac{\partial V}{\partial x}^T\right) - \dot{V}(x(t)). \end{aligned}$$

Since $u(\cdot)$ stabilizes (7) in predefined time T_c, using (18) and (19) we have

$$\begin{aligned} J(x_0,u(\cdot)) &= \int_0^\infty \left[\mathcal{H}\left(x(t),u(t),\frac{\partial V}{\partial x}^T\right) - \dot{V}(x(t))\right]\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) + \\ &\quad \int_0^\infty \mathcal{H}\left(x(t),u(t),\frac{\partial V}{\partial x}^T\right)\mathrm{d}t \\ &= V(x_0) + \int_0^\infty \mathcal{H}\left(x(t),u(t),\frac{\partial V}{\partial x}^T\right)\mathrm{d}t \\ &\ge V(x_0) \\ &= J(x_0,\phi^*(x(\cdot))). \end{aligned}$$

□

Remark 3.1. The conditions (18) and (19) together are exactly the steady-state Hamilton-Jacobi-Bellman equation (12).

Remark 3.2. It is important that the optimal predefined-time stabilizing controller $u^* = \phi^*(x)$ characterized by Theorem 3.1 is a feedback controller.

Although Theorem 3.1 provides sufficient conditions for a controller to solve the optimal predefined-time stabilization problem for a given system, it does not provide a closed form expression for the feedback controller. Instead, the feedback controller is obtained by solving (20). To obtain a closed form expression for the controller, the result of Theorem 3.1 is specialized to affine systems of the form

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t), \qquad x(0) = x_0, \tag{24}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function with $f(0) = 0$ and $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$.

The performance integrand is also specialized to

$$L(x,u) = L_1(x) + L_2(x)u + u^T R_2(x)u, \tag{25}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a positive definite matrix function.

The following corollary of Theorem 3.1 provides an *inverse optimal controller* which solves the optimal predefined-time stabilization problem for the affine system (24) with performance integrands of the form (25).

Corollary 3.1. Assume there exist a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, and real numbers $T_c > 0$ and $0 < p \le 1$ such that

$$V(0) = 0 \tag{26}$$

$$V(x) > 0, \quad \forall x \neq 0, \tag{27}$$

$$\frac{\partial V}{\partial x}\left[f(x) + B(x)\left[-\frac{1}{2}R_2^{-1}(x)\left[L_2(x) + \frac{\partial V}{\partial x}B(x)\right]^T\right]\right] \le -\frac{1}{T_c p}\exp(V^p)V^{1-p} \tag{28}$$

$$L_2(0) = 0 \tag{29}$$

$$L_1(x) + \frac{\partial V}{\partial x}f(x) - \frac{1}{4}\left[L_2(x) + \frac{\partial V}{\partial x}B(x)\right]R_2^{-1}(x)\left[L_2(x) + \frac{\partial V}{\partial x}B(x)\right]^T = 0. \tag{30}$$

Then, with the feedback control

$$u^* = \phi^*(x) = -\frac{1}{2}R_2^{-1}(x)\left[L_2(x) + \frac{\partial V}{\partial x}B(x)\right]^T, \tag{31}$$

the origin of the closed loop system

$$\dot{x}(t) = f(x(t)) + B(x(t))\phi^*(x(t)) \tag{32}$$

is predefined-time stable with $T_f = T_c$. Moreover, the performance measure $J(x_0, u(\cdot))$ is minimized in the sense of (22) and

$$J(x_0, \phi^*(x(\cdot))) = V(x_0), \tag{33}$$

i.e., the feedback control law (31) solves the optimal predefined-time stabilization problem for the system (24).

Proof. We can see that the hypotheses of Theorem Theorem 3.1 are satisfied. The control law (31) follows from $\frac{\partial}{\partial u}\left[\mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right)\right] = 0$ with $L(x, u)$ specialized to (25). Then, setting $u^* = \phi^*(x)$ as in (31), the conditions (26), (27) and (28) become the hypotheses (14), (15) and (17), respectively. The hypothesis (16) follows from (29).

Since $\phi^*(x)$ satisfies $\frac{\partial}{\partial u}\left[\mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right)\right]_{u=\phi^*(x)} = 0$, and noticing that (30) can be rewritten in terms of $\phi^*(x)$ as

$$L_1(x) + \frac{\partial V}{\partial x}f(x) - \phi^{*T}(x)R_2(x)\phi^*(x) = 0. \tag{34}$$

the hypothesis (18) is directly verified.

Finally, from (18), (31) and the positive definiteness of $R_2(x)$ it follows

$$\begin{aligned}\mathcal{H}\left(x, u, \frac{\partial V}{\partial x}\right) &= L(x, u) + \frac{\partial V}{\partial x}[f(x) + B(x)u] \\ &= L(x, u) + \frac{\partial V}{\partial x}[f(x) + B(x)u] - \\ &\quad L(x, \phi^*(x)) - \frac{\partial V}{\partial x}[f(x) + B(x)\phi^*(x)] \\ &= \left[L_2(x) + \frac{\partial V}{\partial x}B(x)\right](u - \phi^*(x)) + \\ &\quad u^T R_2(x)u - \phi^{*T}(x)R_2(x)\phi^*(x) \\ &= -2\,\phi^{*T}(x)R_2(x)(u - \phi^*(x)) + \\ &\quad u^T R_2(x)u - \phi^{*T}(x)R_2(x)\phi^*(x) \\ &= [u - \phi^*(x)]^T R_2(x)[u - \phi^*(x)] \\ &\ge 0,\end{aligned}$$

which is the hypothesis (19). Applying Theorem 3.1, the result is obtained.

□

Remark 3.3. The feedback controller (31) provided by Corollary 3.1 is an inverse optimal controller in the following sense: instead of solving the steady-state HJB equation directly to minimize some given performance measure, it is defined a family of predefined-time stabilizing controllers that minimize a certain cost function. In this case, one can flexibly specify $L_2(x)$ and $R_2(x)$, while from (34) $L_1(x)$ is parametrized as

$$L_1(x) = \phi^{*T}(x)R_2(x)\phi^*(x) - \frac{\partial V}{\partial x}f(x) \ge 0. \tag{35}$$

Remark 3.4. It is not always easy to satisfy the hypotheses (26)-(30) of Corollary 3.1. However, for affine systems of relative degree one the functions $L_2(x)$ and $R_2(x)$ can be easily chosen to fulfill these conditions.

This motivates the following section.

4. INVERSE OPTIMAL PREDEFINED-TIME STABLE REACHING LAW

In this section, first, some basic concepts corresponding to integral manifolds and sliding mode manifolds are reviewed. For this purpose, consider again the autonomous unforced system (1)

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0,$$

Definition 4.1. (Drakunov and Utkin, 1992) Let $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function, and define the manifold $\mathcal{S} = \{x \in \mathbb{R}^n : \sigma(x) = 0\}$. If for an initial condition $x_0 \in \mathcal{S}$, the solution $x(t, x_0) \in \mathcal{S}$ for all t, the manifold $\mathcal{S}$ is called an *integral manifold*.

Definition 4.2. (Drakunov and Utkin, 1992) If there is a nonempty set $\mathcal{N} \subset \mathbb{R}^n - \mathcal{S}$ such that for every initial condition $x_0 \in \mathcal{N}$, there is a finite time $t_s > 0$ in which the system state reaches the manifold $\mathcal{S}$ then the manifold $\mathcal{S}$ is called an *sliding mode manifold*.

Remark 4.1. A sliding mode on a certain sliding manifold can only appear if f is a non-smooth (possibly discontinuous) function. For this case, the solutions of (1) are understood in the Filippov sense (Filippov, 1988).

With the above definitions, the main objective of the controller is to optimally drive the trajectories of affine system (24) to the set $\mathcal{S}$ in a predefined time. The function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ is selected so that the motion of the system (24) restricted to the sliding manifold $\sigma(x) = 0$ has a desired behavior.

The dynamics of σ are described by

$$\dot{\sigma}(t) = a(x(t)) + G(x(t))u(t), \qquad \sigma(x(0)) = \sigma_0, \tag{36}$$

where $a(x) = \frac{\partial \sigma}{\partial x}f(x)$ and $G(x) = \frac{\partial \sigma}{\partial x}B(x)$.

It is assumed that $\sigma(x)$ is selected such that the matrix $G(x) \in \mathbb{R}^{m\times m}$ has inverse for all $x \in \mathbb{R}^n$. It means that the system (36) has relative degree one.

Now, consider the optimal predefined time stabilization problem (13) for the system (36). The aim is to choose the functions V, L_2 and R_2 such that the hypotheses of Corollary 3.1 are satisfied. To this end, assume that $V(\sigma)$

is a Lyapunov function candidate. Its derivative along the trajectories of the system (36) closed loop with (31)

$$\dot{\sigma} = a(x) + G(x)\phi^*(x),$$

is calculated as

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial \sigma}\left[a(x) + G(x)\phi^*(x)\right] \\ &= \frac{\partial V}{\partial \sigma}\left[a(x) + G(x)\left(-\frac{1}{2}R_2^{-1}(x)L_2^T(x) - \right.\right. \\ &\qquad \left.\left. \frac{1}{2}R_2^{-1}(x)G^T(x)\frac{\partial V}{\partial \sigma}^T\right)\right].\end{aligned}$$

Then, choosing

$$L_2(x) = 2a^T(x)\left[G^{-1}(x)\right]^T R_2(x), \tag{37}$$

$$R_2(x) = \frac{T_c p}{2}\exp(-V^p(\sigma(x)))\left[G^T(x)G(x)\right], \tag{38}$$

the derivative of V becomes

$$\dot{V} = -\frac{1}{T_c p}\exp(V^p)\left|\left|\frac{\partial V}{\partial \sigma}\right|\right|^2.$$

Finally, V must be chosen such that

$$\left|\left|\frac{\partial V}{\partial \sigma}\right|\right|^2 = V^{1-p}. \tag{39}$$

It can easily be checked that

$$V(\sigma) = c^i(\sigma^T\sigma)^i > 0, \quad \forall \sigma \neq 0 \tag{40}$$

with $i = \frac{1}{p+1}$ and $4i^2c = \frac{4c}{(p+1)^2} = 1$, satisfies (39).

Example 4.1. Consider a pendulum system with Coulomb friction

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1) - \frac{V_s}{J}x_2 - \frac{P_s}{J}\text{sign}(x_2) + \frac{1}{J}u,\end{aligned} \tag{41}$$

where x_1 is the angular position, x_2 is the angular velocity, u is the input torque, J is the moment of inertia, g is the gravity acceleration, L is the length of the pendulum, and P_s and V_s are friction constants.

Due to the structure of the model (41), a good candidate for σ is $\sigma(x) = x_2 + kx_1$ with $k > 0$. The dynamics of σ are described by the equation (36) with $a(x) = -\frac{g}{L}\sin(x_1) - \frac{V_s}{J}x_2 - \frac{P_s}{J}\text{sign}(x_2) + kx_2$ and $G(x) = \frac{1}{J}$. The functions V, R_2 and L_2 are selected according to (37)-(40) as

$$\begin{aligned}V(\sigma) &= c^{\frac{1}{p+1}}\sigma^{\frac{2}{p+1}}, \\ R_2(x) &= \frac{T_c p}{2J^2}\exp(-c^{\frac{p}{p+1}}\sigma^{\frac{2p}{p+1}}), \text{ and} \\ L_2(x) &= \frac{T_c p}{J}\exp(-c^{\frac{p}{p+1}}\sigma^{\frac{2p}{p+1}})\left[-\frac{g}{L}\sin(x_1) - \frac{V_s}{J}x_2 - \right. \\ &\qquad \left. \frac{P_s}{J}\text{sign}(x_2) + kx_2\right],\end{aligned}$$

and $u^* = \phi^*(x)$ is implemented as in (31).

Finally, according to (35) the resulting function L_1 is

$$L_1(x) = \frac{2J^2}{4T_c p}\exp(c^{\frac{p}{p+1}}\sigma^{\frac{2p}{p+1}})\left[L_2(x) + \frac{1}{J}\frac{\partial V}{\partial \sigma}\right]^2 - \frac{\partial V}{\partial \sigma}a(x).$$

5. SIMULATION RESULTS

The simulation results of the Example 5.1 are presented in this section. The pendulum parameters are shown in Table 1.

Table 1. Parameters of the pendulum model (41).

Parameter	Values	Unit
M	1	kg
L	1	m
$J = ML^2$	1	$\text{kg} \cdot \text{m}^2$
V_s	0.2	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
P_s	0.5	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$
g	9.8	$\text{m} \cdot \text{s}^{-2}2$

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-3} s. The initial conditions for the system (41) were selected as: $x_1(0) = \pi/2\,\text{rad}$ and $x_2(0) = 0\,\text{rad/s}$. In addition, the controller gains were adjusted to: $T_c = 1$, $k = 2$ and $p = 1/2$.

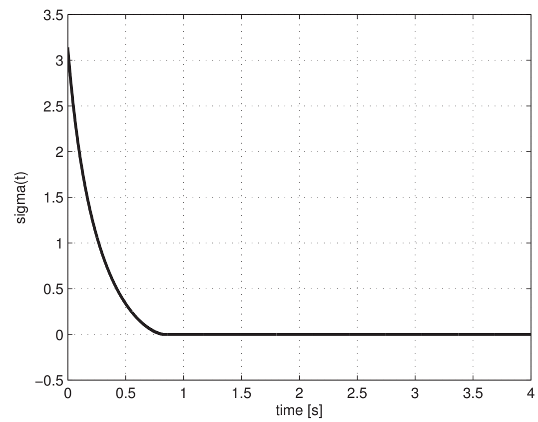

Figure 1. Function $\sigma(x(t))$.

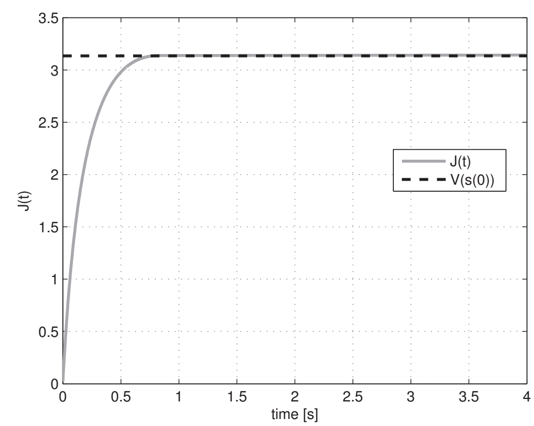

Figure 2. Function $J(t) = \int_0^t [L_1 + L_2 u + u^T R_2 u]\mathrm{d}\tau$.

Note that $\sigma(t) = 0$ for $t \geq 0.827\ s < T_c = 1\ s$ (Fig. 1). Once the system states slide over the sliding manifold $\sigma(x) = 0$, this motion is governed by the reduced order system

$$\dot{x}_1(t) = -kx_1(t) = x_2.$$

This imply that the system state tends exponentially to zero at a rate of $\frac{1}{k}$ (Fig. 3). Fig. 4 shows the control signal (torque) versus time. Finally, from Fig. 2, it can be seen

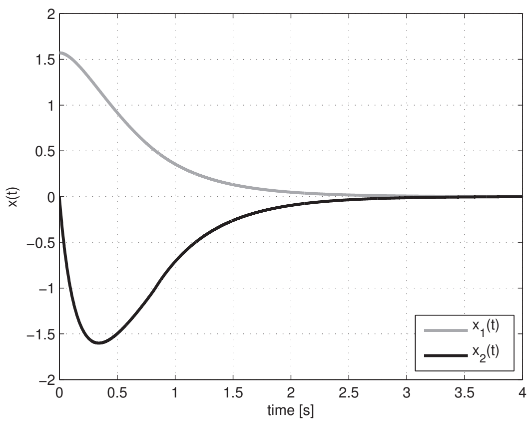

Figure 3. Evolution of the states.

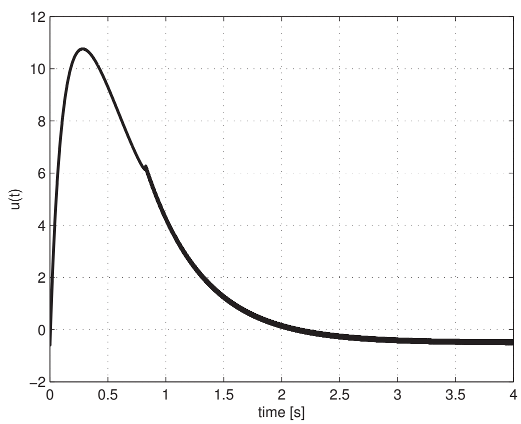

Figure 4. Control input.

that the cost as a function of time grows quickly to a steady state value, corresponding to $V(\sigma(0))$.

6. CONCLUSION

In this paper, the problem of optimal predefined-time stability was addressed. Sufficient conditions for a controller to guarantee both predefined-time stability and optimality were provided. The results were applied to the predefined-time optimization of the sliding manifold reaching phase. This application was illustrated by an example, which was simulated.

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