Predefined-Time Tracking of a Class of Mechanical Systems

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Abstract—In this paper the problem of predefined-time exact tracking of fully actuated and unperturbed mechanical systems is solved by means of a continuous controller. It is assumed the availability of the state and the desired trajectory as well as its two first derivatives. This is accomplished introducing the idea of second-order predefined-time stable systems, which is based on the first-order predefined-time stabilizing function. As an example, the proposed solution is applied over a two-link planar manipulator and numerical simulations are conducted to show its performance.

I. INTRODUCTION

The various developments concerning the concept of finite-time stability permit to solve different applications which are characterized for requiring hard time response constraints. Some important works of this topic and its application to control systems have been carried out in [1]–[6]. However, generally this finite time is an unbounded function of the initial conditions of the system. A desired feature is to eliminate this boundlessness, for example, in estimation or optimization problems. This gives rise to a stronger form of stability called fixed-time stability, where the convergence time, as a function of the initial conditions, is bounded. The notion of fixed-time stability have been investigated in [7]–[11].

Although fixed-time stability represents a significant advantage over finite-time stability, it has two major drawbacks. The first of them is that it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time; the other one is that the bounds of the fixed stabilization time found by Lyapunov analysis constitute usually conservative estimations, i.e. they are much larger than the true fixed stabilization time (see for example [12], where the upper bound estimation is approximately 100 times larger than the actual true fixed stabilization time). To overcome the above, another class of dynamical systems which exhibit the property of predefined-time stability, have been studied [13], [14]. For this systems, the minimum upper bound of the fixed stabilization time appears explicitly in their tuning gains.

Nevertheless, until now, this predefined-time property have been studied only for first-order systems (systems of relative degree one). In this sense, this paper introduces the concept of second-order predefined-time as a nested application of first-order predefined-time stabilizing functions [14]. Furthermore, this idea is used to solve the problem of predefined-time exact tracking in fully actuated mechanical systems, assuming the availability of the state and the desired trajectory (as well as its two first derivatives) measurements.

In the following, Section II presents the mathematical preliminaries needed to introduce the proposed results. Section III states the problem which will be solved in this paper. Section IV exposes the main result of this paper, which is the second-order predefined-time tracking controller for fully actuated mechanical systems. Section V describes the model of a planar two-link manipulator, where the proposed controller is applied. The simulation results of the example are shown in Section VI. Finally, Section VII presents the conclusions of this paper.

II. MATHEMATICAL PRELIMINARIES

Consider the system

$$\dot{x} = f(x; \rho)$$

(1)

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The initial conditions of this system are $x(0) = x_0$.

Definition 2.1 (Global finite-time stability [9]): The origin of (1) is globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$.

Definition 2.2 (Fixed-time stability [9]): The origin of (1) is fixed-time stable if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\text{max}} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\text{max}}$.

Remark 2.1: Note that there are several choices for $T_{\text{max}}$. For instance, if the settling-time function is bounded by $T_m$, it is also bounded by $\lambda T_m$ for all $\lambda \geq 1$. This motivates the following definition.

Definition 2.3 (Setting-time set and its minimum bound [13], [14]): Let $\mathcal{T}$ be the set of all the bounds of the settling time function for the system (1), i.e.,

$$\mathcal{T} = \{T_{\text{max}} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\text{max}}\}.$$  (2)

The minimum bound of the settling-time function $T_f$, is defined as:

$$T_f = \inf_{x_0 \in \mathbb{R}^n} \mathcal{T} = \sup_{x_0} T(x_0).$$  (3)
Remark 2.2: In a strict sense, the time $T_f$ can be considered as the true fixed-time in which the system (1) stabilizes.

Definition 2.4 (Predefined-time stability [13]): For the case of fixed time stability when the time $T_f$ defined in (3) can be tuned by a particular selection of the parameters $\rho$ of the system (1), it is said that the origin of the system (1) is predefined-time stable.

Definition 2.5 (Predefined-time stabilizing function [14]): For $x \in \mathbb{R}^n$, the predefined-time stabilizing function is defined as

$$\Phi_p(x; T_c) = \frac{1}{T_c p} \exp\left(\frac{\|x\|^p}{2} \right) x$$  \hspace{1cm} (4)

where $0 < p \leq 1$ and $T_c > 0$.

Remark 2.3: Since $\lim_{x \to 0} \Phi_p(x; T_c) = 0$ for $0 < p \leq 1$, it is considered that $\Phi_p(0; T_c) = 0$. Therefore, the function defined in (4) is continuous for $0 < p \leq 1$ and discontinuous in $x = 0$ for $p = 1$.

Remark 2.4 (Predefined-time stabilizing function derivative): It can be checked that the derivative of the predefined-time stabilizing function is given by

$$\frac{\partial \Phi_p(x; T_c)}{\partial x} = \frac{\exp\left(\frac{\|x\|^p}{2} \right)}{T_c p} \left[ p\frac{x x^T}{\|x\|^2} + \frac{1}{\|x\|^2} \right].$$  \hspace{1cm} (5)

From the Definition 2.5 of the stabilizing function, the following Lemma presents a dynamical system with the predefined-time stability property.

Lemma 2.1 (Predefined-time stable dynamical system [14]): The origin of the system

$$\dot{x} = -\Phi_p(x; T_c)$$  \hspace{1cm} (6)

with $T_c > 0$, and $0 < p \leq 1$ is predefined-time stable with $T_f = T_c$. That is, $x(t) = 0$ for $t > T_c$ in spite of the $x_0$ value.

Remark 2.5: From (5), the time derivative of the function $\Phi_p(x; T_c)$ defined in (4) along the trajectories of the system (6) is

$$\frac{d \Phi_p(x; T_c)}{dt} = -\frac{\partial \Phi_p(x; T_c)}{\partial x} \Phi_p(x; T_c) = -\frac{\exp\left(\frac{\|x\|^p}{2} \right)}{T_c p} \left[ p\frac{x x^T}{\|x\|^2} + \frac{x}{\|x\|^2} \right].$$

Since $\lim_{x \to 0} \frac{d \Phi_p(x; T_c)}{dt} = 0$ for $0 < p < \frac{1}{2}$, it is considered that $\frac{d \Phi_p(x; T_c)}{dt}$ is continuous for $0 < p < \frac{1}{2}$.

III. PROBLEM STATEMENT

A generic model of second-order, fully actuated mechanical systems of $n$ degrees of freedom has the form

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + P(q) + \gamma(q) = \tau,$$  \hspace{1cm} (7)

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position, velocity and acceleration vectors in joint space; $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis and centrifugal effects matrix, $P(q) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction and $\gamma(q) \in \mathbb{R}^n$ is the gravity effects vector.

Defining the variables $x_1 = q, x_2 = \dot{q}$ and $u = \tau$, the mechanical model (7) can be rewritten in the following state-space form

$$\dot{x}_1 = x_2,$$  \hspace{1cm} (8)

$$\dot{x}_2 = f(x_1, x_2) + B(x_1, x_2)u,$$

where $f(x_1, x_2) = -M^{-1}(x_1) \left[ C(x_1, x_2) x_2 + P(x_2) + \gamma(x_1) \right]$, $B(x_1, x_2) = M^{-1}(x_1)$ are continuous maps and the initial conditions are $x_1(0) = x_{1,0}$, $x_2(0) = x_{2,0}$.

Remark 3.1: The matrix function $M(x_1)$ is, in fact, invertible since $M(x_1) = M^T(x_1)$ is positive definite.

A common problem in mechanical systems control is to track a desired time-dependent trajectory described by the triplet $(q_d(t), \dot{q}_d(t), \ddot{q}_d(t))$ of desired position $q_d(t) = [q_{d1}(t) \ldots q_{dn}(t)]^T \in \mathbb{R}^n$, velocity $\dot{q}_d(t) = [\dot{q}_{d1}(t) \ldots \dot{q}_{dn}(t)]^T \in \mathbb{R}^n$, and acceleration $\ddot{q}_d(t) = [\ddot{q}_{d1}(t) \ldots \ddot{q}_{dn}(t)]^T \in \mathbb{R}^n$, which are all assumed to be known.

To be consequent with the state space notation, the desired position and velocity vectors are redefined as $x_{1,d} = q_d$ and $x_{2,d} = \dot{q}_d = \ddot{x}_{1,d}$, respectively. Then, defining the error variables as $e_1 = x_1 - x_{1,d}$ (position error) and $e_2 = x_2 - x_{2,d}$ (velocity error), the error dynamics are:

$$\dot{e}_1 = e_2,$$  \hspace{1cm} (9)

$$\dot{e}_2 = f(x_1, x_2) + B(x_1, x_2)u - \ddot{x}_{1,d},$$

with initial conditions $e_1(0) = e_{1,0}, x_{1,0} - x_{1,d}(0)$, $e_2(0) = e_{2,0}, x_{2,0} - x_{2,d}(0)$.

The task is to design a state-feedback, second-order, predefined-time controller to track the desired trajectory. In other words, the error variables $e_1$ and $e_2$ are to be stabilized in predefined time with available measurements of $x_1, x_2, x_{1,d}, x_{2,d} = \ddot{x}_{1,d}$ and $\dddot{x}_{1,d}$.

IV. SECOND-ORDER PREDEFINED-TIME TRACKING CONTROLLER

With basis on the function $\Phi_p(x; T_c)$, consider the non-singular transformation

$$\sigma_1 = e_1,$$  \hspace{1cm} (10)

$$\sigma_2 = e_2 + \Phi_p (e_1; T_c_1).$$

where $0 < p_1 < \frac{1}{2}, 0 < p_2 < 1, T_{c1} > 0$, and $T_{c2} > 0$.

From (9), the dynamics of the system in the new coordinates $(\sigma_1, \sigma_2)$ are in a block controllable form (see [15]):

$$\dot{\sigma}_1 = \sigma_2 - \Phi_p (\sigma_1; T_c_1),$$

$$\dot{\sigma}_2 = f(x_1, x_2) + B(x_1, x_2)u - \ddot{x}_{1,d} + \frac{\partial \Phi_p}{\partial \sigma_1} (\sigma_1; T_c_1) \left[ \sigma_2 - \Phi_p (\sigma_1; T_c_1) \right].$$  \hspace{1cm} (11)

with initial conditions $\sigma_1(0) = \sigma_{1,0} = e_{1,0}, \sigma_2(0) = \sigma_{2,0} = e_{2,0} + \Phi_p (e_{1,0}; T_c_1)$.
Hence, for the system (11) the following controller is proposed:

\[
    u = -B^{-1}(x_1, x_2) \left[ f(x_1, x_2) - \ddot{x}_{1,d} + \frac{\partial \Phi_p_1(\sigma_1; T_{c_1})}{\partial \sigma_1} \left[ \sigma_2 - \Phi_p_1(\sigma_1; T_{c_1}) + \Phi_p_2(\sigma_2; T_{c_2}) \right] \right].
\]

Thus, the system (11) closed-loop with the controller (12) has the form:

\[
    \dot{\sigma}_1 = -\Phi_p_1(\sigma_1; T_{c_1}) + \sigma_2, \\
    \dot{\sigma}_2 = -\Phi_p_2(\sigma_2; T_{c_2}).
\]

Taking into account the structure of the system (13), the following theorem states the tracking of the system (7).

**Theorem 4.1**: For the system (7), \( q = q_d \) and \( \dot{q} = \dot{q}_d \) for \( t > T_{c_1} + T_{c_2} \).

**Proof.** To prove Theorem 4.1 it is sufficient to analyze the stability of the system (13). Using Lemma 2.1, \( \sigma_2(t) = 0 \) for \( t > T_{c_2} \), in spite of the initial conditions \( \sigma_{2,0} \). The motion of the system on the manifold \( \sigma_2 = 0 \) is given by \( \dot{\sigma}_1 = -\Phi_p_1(\sigma_1; T_{c_1}) \). Applying again Lemma 2.1, \( \sigma_1(t) = 0 \) for \( t > T_{c_1} + T_{c_2} \), in spite of the value of \( \sigma_1(T_{c_2}) \). Finally, from (10), \( e_1 = e_2 = 0 \) for \( t > T_{c_1} + T_{c_2} \), which directly imply the result.

**Remark 4.1**: The control \( u \) can be written with respect to the parameters \( T_{c_1} \) and \( T_{c_2} \) as:

\[
    u = \begin{cases} 
    u_{\sigma_2=0} & \text{for } t > T_{c_2} \\
    u_{\sigma_1=0,\sigma_2=0} & \text{for } t > T_{c_1} + T_{c_2} 
    \end{cases}
\]

where \( u_{\sigma_2=0} = -B^{-1}(x_1, x_2) \left[ f(x_1, x_2) - \ddot{x}_{1,d} - \left[ \left[ \exp(\|\sigma_1\|) \right]^{-1} \right] \left[ p_1 \frac{\sigma_1}{\|\sigma_1\|} + (1 - p_1) \frac{\sigma_1}{\|\sigma_1\|} \right] \right] \), which is well defined since \( 0 < p_1 < \frac{1}{2} \) (see Remark 2.5), and \( u_{\sigma_1=0,\sigma_2=0} = -B^{-1}(x_1, x_2, d) \left[ f(x_1, x_d, d) - \ddot{x}_{1,d} \right] \).

Therefore, the control input (12) is a continuous function of \( \sigma_1 \) and \( \sigma_2 \).

**V. EXAMPLE: TRAJECTORY TRACKING FOR A TWO-LINK MANIPULATOR**

Consider a planar, two-link manipulator with revolute joints as the one exposed in [16] (see Fig. 1). The manipulator link lengths are \( L_1 \) and \( L_2 \), the link masses (concentrated in the end of each link) are \( M_1 \) and \( M_2 \). The manipulator is operated in the plane, such that the gravity acts along the z-axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass \( M_2 \) is concentrated) position \((x_w, y_w)\) is given by:

\[
    x_w = L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) \\
    y_w = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2),
\]

where \( q_1 \) and \( q_2 \) are the joint positions (angular positions).

Applying the Euler-Lagrange equations, a model according to (7) is obtained, with:

\[
    m_{11} = L_1^2(M_1 + M_2) + 2(L_2^2M_2 + L_1L_1 + M_2 \cos q_2) - L_2^2M_2 \\
    m_{12} = m_{21} = L_2^2M_2 + L_1L_1 \cos q_2 \\
    m_{22} = L_2^2M_2 \\
    h = L_1L_2M_2 \sin q_2 \\
    c_{11} = -h q_2 \\
    c_{12} = -h(q_1 + \dot{q}_2) \\
    c_{21} = h \dot{q}_1 \\
    c_{22} = 0
\]

\[
    M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \\
    C(q, \dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\
    P(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
    \gamma(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The absence of gravity term is because the manipulator is operated in the plane, perpendicular to gravity. Note also that friction terms are neglected.

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius \( r_d \) and center in the origin. To solve this problem the controller exposed in Section IV is applied.

**VI. SIMULATION RESULTS**

The simulation results of the example in Section V are presented in this section. The two-link manipulator parameters used are shown in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>0.2</td>
<td>kg</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>0.2</td>
<td>kg</td>
</tr>
<tr>
<td>( L_1 )</td>
<td>0.2</td>
<td>m</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>0.2</td>
<td>m</td>
</tr>
</tbody>
</table>

The simulations were conducted using the Euler integration method, with a fundamental step size of \( 1 \times 10^{-4} \) s. The
initial conditions for the two-link manipulator were selected as: \( x_1(0) = \left[ -\frac{3\pi}{4} - \frac{\pi}{2} \right]^T \) and \( x_2(0) = [0 \ 0]^T \). In addition, the controller gains were adjusted to: \( T_{c1} = 1, T_{c2} = 0.5, p_1 = \frac{1}{3} \) and \( p_2 = \frac{1}{2} \).

The desired circular trajectory in the joint coordinates is described by the equations

\[
q_d(t) = x_1,d(t) = \begin{bmatrix} q_{d1}(t) \\ q_{d2}(t) \end{bmatrix} = \begin{bmatrix} \frac{2\pi t}{\tau} - \frac{\pi}{\tau} \\ -\frac{\pi}{\tau} \end{bmatrix}, \quad (15)
\]

and it corresponds to a circumference of radius 0.2828 m.

The following figures show the behavior of the proposed controller.

![Fig. 2. Variable \( \sigma_2 \). First component (gray and solid) and second component (black and dashed). Note that \( \sigma_2(t) = 0 \) for \( t > T_{c2} = 0.5 \) s.](image2.png)

![Fig. 3. Error variables. First component of \( e_1 \) (dark gray and thick), second component of \( e_1 \) (black and dashed), first component of \( e_2 \) (light gray and solid) and second component of \( e_2 \) (black and solid).](image3.png)

![Fig. 4. Control signal. First component (gray and solid) and second component (black and solid).](image4.png)

![Fig. 5. Actual trajectory \((x_{w,d}, y_{w,d})\) (black and solid) and desired trajectory \((x_{w,d}, y_{w,d})\) (black and dashed). Note that \( \sigma_2(t) = 0 \) for \( t > 0.47 \) s < \( T_{c2} = 0.5 \) s (Fig. 2). Once the error variables slide over the manifold \( \sigma_2 = 0 \), this motion is governed by the reduced order system

\[
\dot{e}_1 = e_2 = -\Phi_{p1}(e_1; T_{c1})).
\]

This imply that the error variables are exactly zero for \( t > T_{c1} + T_{c2} = 1.5 \) s. In fact, from Fig. 3, it can be seen that \( e_1(t) = e_2 = 0 \) for \( t \geq 0.74 \) s < \( T_{c1} + T_{c2} = 1.5 \) s. Fig. 4 shows the control signal (torque) versus time. Finally, from Fig. 5, it can be seen the reference tracking in rectangular coordinates.

VII. CONCLUSION

In this paper the problem of predefined-time exact tracking in fully actuated mechanical systems was solved by means of a controller which induces second-order predefined-time stability in the tracking error. This controller was constructed as a nested application of continuous first-order predefined-time stabilizing functions.

To show the feasibility of the proposed controller, it was implemented over a two-link planar manipulator. The numerical simulations showed a good performance.
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