# Predefined-Time Stabilization of High Order Systems*

Esteban Jiménez-Rodríguez[1], Juan Diego Sánchez-Torres[2], David Gómez-Gutiérrez[1] and Alexander G. Loukianov[1]

***Abstract*— The aim of this paper is to introduce a controller that stabilizes a class of arbitrary order systems in predefined-time. The proposed controller is designed with basis on the block-control principle yielding in a nested structure similar to high order sliding mode algorithms and terminal sliding mode algorithms. For this case, it is assumed the availability of the state and the absence of perturbations. Numerical simulations expose the desired performance of this controller.**

## I. INTRODUCTION

Several applications are characterized for requiring hard time response constraints. In order to deal with those requirements, various developments concerning to concept of *finite-time stability* have been carried out (see for example: [1]–[8]). Nevertheless, usually this finite time is an unbounded function of the initial conditions of the system.

With the aim to eliminate this boundlessness, the notion of fixed-time stability have been studied in [9]–[13]. Fixed-time stability represents a significant advantage over finite-time stability due to its desired feature of the convergence time, as a function of the initial conditions, is bounded. That makes the fixed-time stability a valuable feature in estimation and optimization problems.

For the most of the proposed fixed-time stable system, there are problems related with the convergence time. First, the bounds of the fixed stabilization time found by Lyapunov analysis constitute usually conservative estimations, i.e. they are much larger than the true fixed stabilization time (see for example [14], where the upper bound estimation is approximately 100 times larger than the actual true fixed stabilization time). Second, and as consequence, it is often complicated to find a direct relationship between the tuning gains and the fixed stabilization time, making this time hard to tune.

To overcome the above, a class of first-order dynamical systems with the minimum upper bound of the fixed stabilization time equal to their only tuning gain has been studied [15], [16], similarly a class of second-order systems with similar features is presented in [17]. It is said that these systems exhibit the property of *predefined-time stability*.

In this sense, this paper extends the concept of predefined-time stability to higher-order systems by designing a controller which induces this form of stability to a class of arbitrary order systems. Similarly to [17], the proposed scheme allows to define high-order predefined-time stable systems as a nested application of first-order predefined-time stabilizing functions.

In the following, Section II presents the mathematical preliminaries needed to introduce the proposed results. Section III exposes the main result of this paper, which is the a controller that stabilizes an arbitrary order system in predefined-time. Section IV describes a third-order systems where the proposed controller is applied. In addition, the simulation results of the example are shown. Section V presents the conclusions of this paper. Finally, an Appendix with the more sophisticated proofs is added after the conclusions.

## II. MATHEMATICAL PRELIMINARIES

### A. On predefined-time stability

Consider the system

$$\dot{x} = f(x;\rho) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\rho \in \mathbb{R}^b$ represents the parameters of the system and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The initial conditions of this system are $x(0) = x_0$.

*Definition 2.1 ( [11]):* The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(x_0) : x(t, x_0) = 0$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$.

*Definition 2.2 ( [11]):* The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\text{max}} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\text{max}}$.

*Remark 2.1:* Note that there are several choices for $T_{\text{max}}$. For instance, if the settling-time function is bounded by $T_{\text{m}}$, it is also bounded by $\lambda T_{\text{m}}$ for all $\lambda \geq 1$. This motivates the following definition.

*Definition 2.3 ( [15], [16]):* Let $\mathcal{T}$ be the set of all the bounds of the settling time function for the system (1), i.e.,

$$\mathcal{T} = \left\{T_{\text{max}} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\text{max}}\right\}. \tag{2}$$

The minimum bound of the settling-time function $T_f$, is defined as:

$$T_f = \inf \mathcal{T} = \sup_{x_0 \in \mathbb{R}^n} T(x_0). \tag{3}$$

*Remark 2.2:* In a strict sense, the time $T_f$ can be considered as the true fixed-time in which the system (1) stabilizes.

*This work was supported by CONACyT, México, under grant 252405.

[1]Department of Electrical Engineering, CINVESTAV-IPN Guadalajara, Av. del Bosque 1145 Col. El Bajío CP 45019, México. [ejimenezr,dgomez,louk]@gdl.cinvestav.mx

[2]Department of Mathematics and Physics, ITESO, Periférico Sur Manuel Gómez Morín 8585 C.P. 45604, Tlaquepaque, Jalisco, México. dsanchez@iteso.mx

*Definition 2.4 ( [15], [16]):* For the case of fixed time stability when the time $T_f$ defined in (3) can be tuned by a particular selection of the parameters $\rho$ of the system (1), it is said that the origin of the system (1) is *predefined-time stable*.

### B. On predefined-time stabilization

*Definition 2.5:* Let $h \geq 0$. For $x \in \mathbb{R}$, define the function

$$\lfloor x \rceil^h = |x|^h \operatorname{sign}(x),$$

with $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(0) \in [-1, 1]$. Moreover, if $h > 0$, it is defined $\lfloor 0 \rceil^h = 0$.

*Remark 2.3:* For $x \in \mathbb{R}$, some properties of the function $\lfloor \cdot \rceil^h$ are:

*(i)* $\lfloor x \rceil^h$ is continuous for $h > 0$.
*(ii)* $\lfloor x \rceil^0 = \operatorname{sign}(x)$.
*(iii)* $\lfloor x \rceil^1 = \lfloor x \rceil = x$,
*(iv)* $\lfloor 0 \rceil^h = 0$ for $h > 0$.
*(v)* $\frac{d|x|^h}{dx} = h \lfloor x \rceil^{h-1}$ and $\frac{d\lfloor x \rceil^h}{dx} = h |x|^{h-1}$.
*(vi)* For $h_1, h_2 \in \mathbb{R}$, it follows:
   - $\lfloor x \rceil^{h_1} |x|^{h_2} = |x|^{h_1} \lfloor x \rceil^{h_2} = \lfloor x \rceil^{h_1+h_2}$
   - $\lfloor x \rceil^{h_1} \lfloor x \rceil^{h_2} = |x|^{h_1+h_2}$
*(vii)* For $h_1, h_2 > 0$, then $\left\lfloor \lfloor x \rceil^{h_1} \right\rceil^{h_2} = \lfloor x \rceil^{h_1 h_2}$.

*Definition 2.6 ( [15], [16]):* For $x \in \mathbb{R}$, the *predefined-time stabilizing function* is defined as:

$$\Phi_p(x; T_c) = \frac{1}{T_c p} \exp\left(|x|^p\right) \lfloor x \rceil^{1-p} \tag{4}$$

where $T_c > 0$ and $0 < p \leq 1$.

*Remark 2.4:* As a consequence of annotations *(i)* and *(ii)* in Remark 2.3, the function (4) is continuous for $p < 1$ and discontinuous for $p = 1$.

From Definition 2.6, the following Lemma presents a dynamical system with the predefined-time stability property, giving meaning to the name of the function (4).

*Lemma 2.1 ( [15], [16]):* The origin of the system

$$\dot{x} = -\Phi_p(x; T_c) \tag{5}$$

with $T_c > 0$, and $0 < p \leq 1$ is predefined-time stable with $T_f = T_c$. That is, $x(t) = 0$ for $t > T_c$ in spite of the $x(0)$ value.

The following lemma characterizes the form of the high-order derivatives of the function (4).

*Lemma 2.2:* Let $m \in \mathbb{N}$. The $m$-th order derivative of the predefined-time stabilizing function (4) has the form

$$\frac{d^m \Phi_p(x; T_c)}{dx^m} = \frac{1}{T_c p} \exp\left(|x|^p\right) \mathcal{P}_m^s(|x|^p) \lfloor x \rceil^{-p-(m-1)} \tag{6}$$

where $x \neq 0$ and $\mathcal{P}_m^s(|x|^p) = \sum_{j=0}^m c_j^m |x|^{jp} (\operatorname{sign}(x))^m$, with $c_{m+1}^{m+1} = p c_m^m$, $c_i^{m+1} = p c_{j-1}^m + c_j^m ((j-1)p - (m-1))$ for $i = 1, \ldots, m$, $c_0^{m+1} = c_0^m (-p - (m-1))$ and, $c_1^1 = p$ and $c_0^1 = 1 - p$.

## III. HIGH ORDER PREDEFINED-TIME STABILIZING FEEDBACK CONTROLLER

### A. Problem statement

Consider the $n$-th order scalar system in canonical form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_k &= x_{k+1} \text{ for } k = 3, \ldots, n-1 \\
\dot{x}_n &= f(x) + b(x)u,
\end{aligned} \tag{7}$$

where $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$ are the state variables, $x = [x_1 \quad \cdots \quad x_n]^T \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the input control signal. Furthermore, $f : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are nonlinear functions, and $b(x) \neq 0$ for all $x \in \mathbb{R}^n$. The initial conditions of this system are $x(0) = [x_1(0) \quad \cdots \quad x_n(0)]^T = [x_{1,0} \quad \cdots \quad x_{n,0}]^T = x_0$.

The task is to design a predefined-time stabilizing feedback controller for the system (7). In other words, the state $x$ is to be stabilized in predefined time assuming that it is available for measurement.

### B. Controller design

With basis on the function $\Phi_p(x; T_c)$, regard the non-singular transformation

$$\begin{aligned}
\sigma_1 &= x_1 \\
\sigma_2 &= x_2 + \Phi_{p_1}(\sigma_1; T_{c_1}) \\
\sigma_3 &= x_3 + \frac{d\Phi_{p_1}(\sigma_1; T_{c_1})}{dt} + \Phi_{p_2}(\sigma_2; T_{c_2}) \\
\sigma_k &= x_k + \sum_{j=1}^{k-2} \frac{d^{k-1-j} \Phi_{p_j}(\sigma_j; T_{c_j})}{dt^{k-1-j}} \\
&\quad + \Phi_{p_{k-1}}(\sigma_{k-1}; T_{c_{k-1}})
\end{aligned} \tag{8}$$

with $k = 4, \ldots, n$, where $0 < p_i < \frac{1}{n-i+1}$ and $T_{c_i} > 0$ for $i = 1, \ldots, n-1$. Furthermore, for $m \in \mathbb{N}$, $\frac{d^m(\cdot)}{dt^m}$ denotes the $m$-th order derivative of the argument $(\cdot)$ with respect to time.

Applying the transformation (8), the system (7) can be presented in the coordinates $(\sigma_1, \sigma_2 \ldots, \sigma_n)$ as

$$\begin{aligned}
\dot{\sigma}_1 &= -\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2 \\
\dot{\sigma}_2 &= -\Phi_{p_2}(\sigma_2; T_{c_2}) + \sigma_3 \\
\dot{\sigma}_k &= -\Phi_{p_k}(\sigma_k; T_{c_k}) + \sigma_{k+1} \text{ for } k = 3, \ldots, n-1 \\
\dot{\sigma}_n &= f(x) + b(x)u + \sum_{j=1}^{n-1} \frac{d^{n-j} \Phi_{p_j}(\sigma_j; T_{c_j})}{dt^{n-j}},
\end{aligned} \tag{9}$$

with initial conditions $\sigma_1(0) = \sigma_{1,0}, \ \ldots, \ \sigma_n(0) = \sigma_{n,0}$, obtained evaluating $t = 0$ in (8).

Hence, selecting the control input $u$ as

$$u = -\frac{1}{b(x)} [f(x) + v] \tag{10}$$

where $v = \sum_{j=1}^{n-1} \frac{d^{n-j} \Phi_{p_j}(\sigma_j; T_{c_j})}{dt^{n-j}} + \Phi_{p_n}(\sigma_n; T_{c_n})$ with $T_{c_n} > 0$ and $0 < p_n \leq 1$, the system (9) becomes

$$\begin{aligned}
\dot{\sigma}_k &= -\Phi_{p_k}(\sigma_k; T_{c_k}) + \sigma_{k+1} \text{ for } k = 1, \ldots, n-1 \\
\dot{\sigma}_n &= -\Phi_{p_n}(\sigma_n; T_{c_n}).
\end{aligned} \tag{11}$$

*Example 3.1:* The following are expressions of the controller (10) for $n = 1, 2, 3$.

- For $n = 1$
$$u = -\frac{1}{b(x)}\left[f(x) + \Phi_{p_1}(\sigma_1; T_{c_1})\right],$$
with $\sigma_1 = x_1$ as in (8), $p_1 \leq 1$ and $T_{c_1} > 0$.
- For $n = 2$
$$u = -\frac{1}{b(x)}\Bigg[f(x) + \frac{d\Phi_{p_1}(\sigma_1; T_{c_1})}{d\sigma_1}\big(-\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2\big) + \Phi_{p_2}(\sigma_2; T_{c_2})\Bigg],$$
with $\sigma_1 = x_1$ and $\sigma_2 = x_2 + \Phi_{p_1}(\sigma_1; T_{c_1})$ as in (8), $p_1 < \frac{1}{2}$, $p_2 \leq 1$ and $T_{c_1}, T_{c_2} > 0$.
- For $n = 3$
$$u = -\frac{1}{b(x)}\Bigg[f(x) + \frac{d^2\Phi_{p_1}(\sigma_1; T_{c_1})}{d\sigma_1^2}\big(-\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2\big)^2 - \left(\frac{d\Phi_{p_1}(\sigma_1; T_{c_1})}{d\sigma_1}\right)^2\big(-\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2\big) + \frac{d\Phi_{p_1}(\sigma_1; T_{c_1})}{d\sigma_1}\big(-\Phi_{p_2}(\sigma_2; T_{c_2}) + \sigma_3\big) + \frac{d\Phi_{p_2}(\sigma_2; T_{c_2})}{d\sigma_2}\big(-\Phi_{p_2}(\sigma_2; T_{c_2}) + \sigma_3\big) + \Phi_{p_3}(\sigma_3; T_{c_3})\Bigg],$$
with $\sigma_1 = x_1$, $\sigma_2 = x_2 + \Phi_{p_1}(\sigma_1; T_{c_1})$ and $\sigma_3 = x_3 + \frac{d\Phi_{p_1}(\sigma_1; T_{c_1})}{d\sigma_1}\big(-\Phi_{p_1}(\sigma_1; T_{c_1}) + \sigma_2\big) + \Phi_{p_2}(\sigma_2; T_{c_2})$ as in (8), $p_1 < \frac{1}{3}$, $p_2 < \frac{1}{2}$, $p_3 \leq 1$ and $T_{c_1}, T_{c_2}, T_{c_3} > 0$.

In the above expressions
$$\Phi_{p_i}(\sigma_i; T_{c_i}) = \frac{1}{T_{c_i} p_i}\exp(|\sigma_i|^{p_i})\lfloor\sigma_i\rceil^{1-p_i} \quad \text{for } i = 1, 2, 3$$
$$\frac{d\Phi_{p_i}(\sigma_i; T_{c_i})}{d\sigma_i} = \frac{1}{T_{c_i} p_i}\exp(|\sigma_i|^{p_i})\mathcal{P}_1^s(|\sigma_i|^{p_i})\lfloor\sigma_i\rceil^{-p_i} \quad \text{for } i = 1, 2$$
$$\frac{d^2\Phi_{p_1}(\sigma_1; T_{c_1})}{d^2\sigma_1} = \frac{1}{T_{c_1} p_1}\exp(|\sigma_1|^{p_1})\mathcal{P}_2^s(|\sigma_1|^{p_1})\lfloor\sigma_1\rceil^{-1-p_1},$$
where $\mathcal{P}_1^s(|\sigma_i|^{p_i}) = [p_i|\sigma_i|^{p_i} + (1 - p_i)]\operatorname{sign}(\sigma_i)$ and $\mathcal{P}_2^s(|\sigma_1|^{p_1}) = p_1^2|\sigma_1|^{2p_1} + p_1(1 - p_1)|\sigma_1|^{p_1} - p_1(1 - p_1)$.

## C. Stability analysis

The predefined-time stability of the closed-loop system (11) is a direct consequence of Lemma 2.1, and is stated in the following theorem.

*Theorem 3.1:* For the system (7) closed-loop with the controller (10), $x_i(t) = 0$ for $i = 1, \ldots, n$ and $t > \sum_{j=1}^n T_{c_j}$.

*Proof:* To prove Theorem 3.1 it is sufficient to analyze the stability of the system (11).

Using Lemma 2.1, $\sigma_n(t) = 0$ for $t > T_{c_n}$, in spite of the initial condition $\sigma_{n,0}$. The motion of the system on the manifold $\sigma_n = 0$ is given by
$$\begin{aligned}\dot{\sigma}_k &= -\Phi_{p_k}(\sigma_k; T_{c_k}) + \sigma_{k+1} \text{ for } k = 1, \ldots, n-2\\ \dot{\sigma}_{n-1} &= -\Phi_{p_{n-1}}(\sigma_{n-1}; T_{c_{n-1}}).\end{aligned}$$
Applying Lemma 2.1 again, $\sigma_{n-1}(t) = 0$ for $t > T_{c_{n-1}} + T_{c_n}$, in spite of the initial condition $\sigma_{n-1,0}$.

Assume now $\sigma_n = 0$, $\sigma_{n-1} = 0$, …, $\sigma_{l+1} = 0$, with $l \in \mathbb{N}$ and $l \leq n - 1$. The motion on this manifold (for $t > \sum_{j=l+1}^n T_{c_j}$) is given by
$$\begin{aligned}\dot{\sigma}_k &= -\Phi_{p_k}(\sigma_k; T_{c_k}) + \sigma_{k+1} \text{ for } k = 1, \ldots, l-1\\ \dot{\sigma}_l &= -\Phi_{p_l}(\sigma_l; T_{c_l}).\end{aligned}$$
Applying Lemma 2.1 one more time, $\sigma_l(t) = 0$ for $t > \sum_{j=l}^n T_{c_j}$, in spite of the initial condition $\sigma_{l,0}$.

Then, $\sigma_i(t) = 0$ for $i = 1, \ldots, n$ and $t > \sum_{j=1}^n T_{c_j}$, and finally, from the transformation equation (8), $x_i(t) = 0$ for $i = 1, \ldots, n$ and $t > \sum_{j=1}^n T_{c_j}$.

■

## D. Additional remarks

Finally, the good definition of the transformation (8) and the controller (10) are verified.

Recall that the high order derivative $\frac{d^m\Phi_p(x; T_c)}{dx^m}$ presents a singularity at the point $x = 0$ (see (6), Lemma 2.2). Hence, it is necessary to verify that the terms $\frac{d^m\Phi_{p_l}(\sigma_l; T_{c_l})}{dt^m}$ for $l, m \in \mathbb{N}$, with $l \leq n-1$ and $m \leq n-l$, are well-defined (do not present singularities at the points $\sigma_l = 0$), since they appear in the transformation (8) and in the controller (10).

On the other hand, due to the nested structure of the system (11), for $\sigma_l$ to be zero, it must be that $\sigma_{l+1} = 0, \ldots, \sigma_n = 0$ before. Consequently, to analyze the term $\frac{d^m\Phi_{p_l}(\sigma_l; T_{c_l})}{dt^m}$ at $\sigma_l = 0$, it is considered that the system is evolving on the manifold $(\sigma_{l+1}, \ldots, \sigma_n) = (0, \ldots, 0)$, i.e., it evolves according to the reduced order system
$$\begin{aligned}\dot{\sigma}_k &= -\Phi_{p_k}(\sigma_k; T_{c_k}) + \sigma_{k+1} \text{ for } k = 1.\ldots, l-1\\ \dot{\sigma}_l &= -\Phi_{p_l}(\sigma_l; T_{c_l}).\end{aligned} \tag{12}$$

The following lemma provides a form of the terms $\frac{d^m\Phi_{p_l}(\sigma_l; T_{c_l})}{dt^m}$.

*Lemma 3.1:* Let $l, m \in \mathbb{N}$, with $l \leq n-1$ and $m \leq n-l$. Then, the $m$-th order derivative with respect to time of the function $\Phi_{p_l}(\sigma_l; T_{c_l})$, along the trajectories of the system (12), has the form
$$\frac{d^m\Phi_{p_l}(\sigma_l; T_{c_l})}{dt^m} = \sum_{k:N_k^m\in I_m} a_k\left(\prod_{j:z_j^k\in N_k^m}\frac{d^{z_j^k}\Phi_{p_l}(\sigma_l; T_{c_l})}{d\sigma_l^{z_j^k}}\right)(\Phi_{p_l}(\sigma_l; T_{c_l}))^{r_k}, \tag{13}$$
where $I_m = \left\{N_k^m \subset \mathbb{N} : \sum_{j:z_j^k\in N_k^m} z_j^k = m\right\}$ is the family of Natural numbers subsets which sum $m$, $q_k = |N_k^m|$ is the cardinal of the set $N_k^m$, $r_k \in \mathbb{N}$ is such that $m+1 = r_k + q_k$ and $a_k \in \mathbb{R}$.

From Lemma 3.1, to verify the good definition, namely the absence of singularities, of the terms

$\frac{d^m \Phi_{p_l}(\sigma_l;T_{c_l})}{dt^m}$ in (13), it suffices to check the terms $\left(\prod_{j:z_j^k\in N_k^m} \frac{d^{z_j^k}\Phi_{p_l}(\sigma_l;T_{c_l})}{d\sigma_l^{z_j^k}}\right)(\Phi_{p_l}(\sigma_l;T_{c_l}))^{r_k}$.

Applying Lemma 2.2, it follows that $\prod_{j:z_j^k\in N_k^m} \frac{d^{z_j^k}\Phi_{p_l}(\sigma_l;T_{c_l})}{d\sigma_l^{z_j^k}} = \beta_k\left(|\sigma_l|^{p_l}\right)\lfloor\sigma_l\rceil^{q_k(1-p_l)-m}$ with $\beta_k\left(|\sigma_l|^{p}\right) = \left(\frac{1}{T_{c_l}p_l}\right)^{q_k}\exp\left(q_k\left|\sigma_l\right|^{p_l}\right)\prod_{j:z_j^k\in N_k^m}\mathcal{P}^s_{z_j^k}\left(|\sigma_l|^p_l\right)$.

Let $\alpha_k\left(|\sigma_l|^p\right) = \left(\frac{1}{T_{c_l}p_l}\right)^{r_k}\exp\left(r_k\left|\sigma_l\right|^{p_l}\right)\beta_k\left(|\sigma_l|^p\right)$, therefore in (13) it can be observed that

$$\left(\prod_{j:z_j^k\in N_k^m} \frac{d^{z_j^k}\Phi_{p_l}(\sigma_l;T_{c_l})}{d\sigma_l^{z_j^k}}\right)(\Phi_{p_l}(\sigma_l;T_{c_l}))^{r_k} \qquad (14)$$
$$= \alpha_k\left(|\sigma_l|^p\right)\left|\sigma_l\right|^{1-(m+1)p_l}.$$

*Remark 3.1:* Since $m \leq n-l$, the selection of $p_l < \frac{1}{n-l+1}$ for $l = 0,\ldots,n-1$ in Transformation (8) ensures its continuity.

*Remark 3.2:* The continuity of all the terms in the controller except $\Phi_{p_n}(\sigma_n;T_{c_n})$ is assured. The controller can either be continuous or discontinuous depending on whether $p_n < 1$ or $p_n = 1$, respectively.

## IV. SIMULATION RESULTS

The simulations results of the high order predefined-time stabilizing controller for the system (7) with $n = 3$, $f(x) = 0$ and $b(x) = 1$ (triple integrator system) are presented in this section.

The simulations were conducted using the Euler integration method, with a fundamental step size of $1 \times 10^{-5}$ s. The initial conditions for the triple integrator system were selected as: $x(0) = [0.5 \quad -0.3 \quad 0.9]^T$. In addition, the controller took the form presented in Example 3.1, with its gains adjusted to: $T_{c_1} = 1.5$, $T_{c_2} = 1$, $T_{c_3} = 0.5$, $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$.

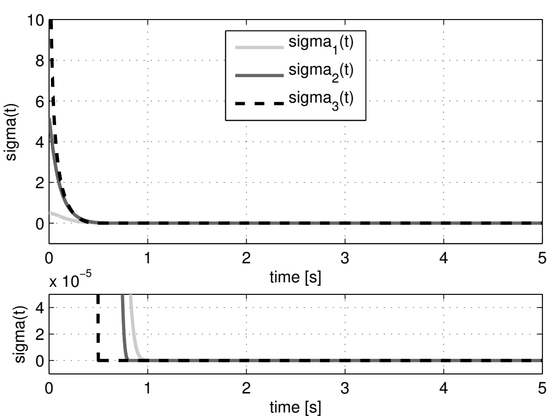


Fig. 1. Transformation variables $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$.

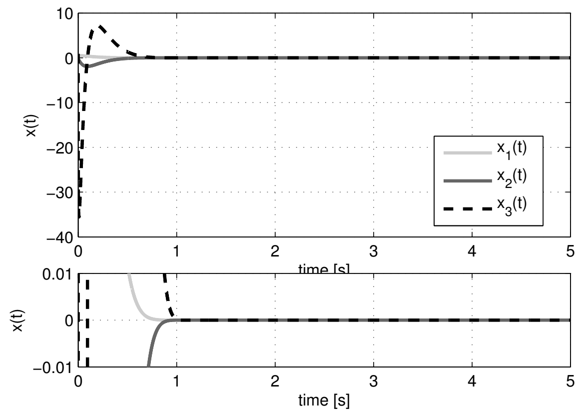


Fig. 2. State variables $x_1(t)$, $x_2(t)$ and $x_3(t)$.

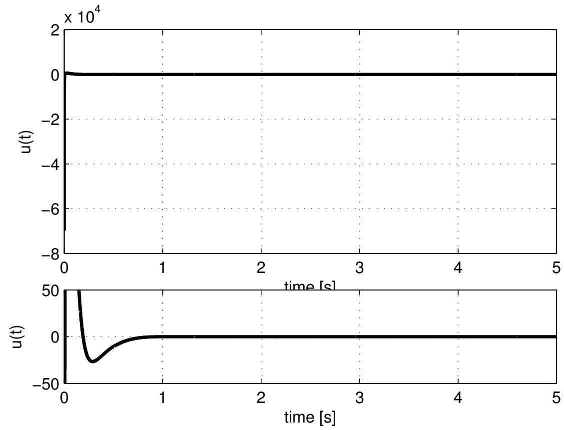


Fig. 3. Control input $u(t)$.

Note that $\sigma_3(t) = 0$ for $t \geq 0.5$ s $< T_{c_3} = 0.5$ s, $\sigma_2(t) = 0$ for $t \geq 0.75$ s $< T_{c_2} + T_{c_3} = 1.5$ s and $\sigma_1(t) = 0$ for $t \geq 0.9$ s $< T_{c_1} + T_{c_2} + T_{c_3} = 3$ s (Fig. 1). It can be noted also that for $t \geq 0.9$ s $< T_{c_1} + T_{c_2} + T_{c_3} = 3$ s (once $\sigma_1(t) = 0$), $x(t) = 0$ as expected (see Fig. 2). Finally, Fig. 3 shows the control signal versus time.

## V. CONCLUSIONS

In this paper a high-order controller with predefined-time stability was introduced. As a result, the trajectories of the controlled dynamical system are driven to the origin in a predefined-time. The presented simulations expose the high performance of the proposal, driving a third-order system to its equilibrium in a defined in advance time.

## APPENDIX

### A. Proof of Lemma 2.2

The proof of this lemma is performed using the induction principle over $m$. The following proposition provides the first-order derivative, which will be used for the base case ($m = 1$).

*Proposition 5.1:* For $x \neq 0$, the first-order derivative of the predefined-time stabilizing function (4) is given by

$$\frac{d\Phi_p(x;T_c)}{dx} = \frac{1}{T_c p}\exp\left(|x|^p\right)\left[p\,|x|^p + (1-p)\right]|x|^{-p}$$

With the statement of the base case, the proof is presented.

*Proof:* (*of Lemma 2.2*) The proof is done using the induction principle.

By Proposition 5.1, (6) is valid for $m = 1$, with $c_1^1 = p$ and $c_0^1 = 1 - p$. Hence, it is assumed that (6) holds for $m$ and checked that it also holds for $m+1$.

The $m$-order derivative (6) can be rewritten as

$$\begin{aligned}&\frac{d^m\Phi_p(x;T_c)}{dx^m} \qquad (15)\\ &= \frac{1}{T_c p}\exp\left(|x|^p\right)\mathcal{P}_m^s(|x|^p)\,\lfloor x\rceil^{-p-(m-1)}\\ &= \frac{1}{T_c p}\exp\left(|x|^p\right)\left[\sum_{j=0}^{m} c_j^m\,|x|^{(j-1)p-(m-1)}\right](\mathrm{sign}(x))^{m+1}.\end{aligned}$$

Then, differentiating (15) it yields the $m+1$ derivative

$$\begin{aligned}&\frac{d^{m+1}\Phi_p(x;T_c)}{dx^{m+1}}\\ &= \frac{1}{T_c p}\exp\left(|x|^p\right)\left[p\,\lfloor x\rceil^{p-1}\sum_{j=0}^{m} c_j^m\,|x|^{(j-1)p-(m-1)} + \right.\\ &\left.\sum_{j=0}^{m} c_j^m\left((j-1)p-(m-1)\right)|x|^{(j-1)p-m}\right](\mathrm{sign}(x))^{m+1}\\ &= \frac{1}{T_c p}\exp\left(|x|^p\right)\left[\sum_{j=0}^{m+1} c_j^{m+1}\,|x|^{(j-1)p-m}\right](\mathrm{sign}(x))^{m+2}\\ &= \frac{1}{T_c p}\exp\left(|x|^p\right)\mathcal{P}_{m+1}^s(|x|^p)\,\lfloor x\rceil^{-p-m}\end{aligned}$$

with $c_{m+1}^{m+1} = pc_m^m$, $c_i^{m+1} = pc_{j-1}^m + c_j^m\left((j-1)p-(m-1)\right)$ for $i = 1,\ldots,m$ and $c_0^{m+1} = c_0^m\left(-p-(m-1)\right)$, completing the proof. ∎

### B. Proof of Lemma 3.1

As in the proof of Lemma 2.2, a the first-order time derivative of the function $\Phi_{p_l}(\sigma_l;T_{c_l})$ is stated in the following proposition.

*Proposition 5.2:* Let $l \in \mathbb{N}$, with $l \leq n-1$. The derivative of the function $\Phi_{p_l}(\sigma_l;T_{c_l})$ with respect to time, along the trajectories of the system (12), is given by

$$\begin{aligned}\frac{d\Phi_{p_l}(\sigma_l;T_{c_l})}{\mathrm{d}t} &= \frac{d\Phi_{p_l}(\sigma_l;T_{c_l})}{d\sigma_l}\dot{\sigma}_l\\ &= -\frac{d\Phi_{p_l}(\sigma_l;T_{c_l})}{d\sigma_l}\Phi_{p_l}(\sigma_l;T_{c_l}).\end{aligned}$$

*Proof:* (*of Lemma 3.1*) By Proposition 5.2, (13) is valid for $m = 1$. In this case $I_1 = \left\{N_1^1\right\}$, $N_1^1 = \left\{z_1^1\right\}$, $z_1^1 = 1$, $a_1 = -1$, $q_1 = \left|N_1^1\right| = 1$ and $r_1 = 1$. Note that $m + 1 = 2 = q_1 + r_1$.

Assume now that (13) is valid for $m < n - l$ and let's see that it also holds for $m+1$. Differentiating (13) with respect to time, it yields:

$$\frac{d^{m+1}\Phi_{p_l}}{\mathrm{d}t^{m+1}} = \sum_k a_k \frac{d}{\mathrm{d}t}\left[\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\Phi_{p_l}^{r_k}\right],$$

where some notation is omitted to ease the reading. Applying the product rule, the above derivative can be expanded as

$$\begin{aligned}\frac{d}{\mathrm{d}t}\left[\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\Phi_{p_l}^{r_k}\right] &= \left[\frac{d}{\mathrm{d}t}\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\Phi_{p_l}^{r_k} + \right.\\ &\left.\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\frac{d\Phi_{p_l}^{r_k}}{\mathrm{d}t}\right].\end{aligned}$$

The first term can be written as follows:

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\Phi_{p_l}^{r_k} =\\ &\qquad -\sum_j \frac{d^{z_j^k+1}\Phi_{p_l}}{d\sigma_l^{z_j^k+1}}\prod_{i\neq j}\frac{d^{z_i^k}\Phi_{p_l}}{d\sigma_l^{z_i^k}}\Phi_{p_l}^{r_k+1}.\end{aligned}$$

Clearly, each term of the sum has a product of $q_k$ derivatives and, since we are evaluating for $m+1$, the following equation holds $(m+1) - (r_k+1) + 1 = m - r_k + 1 = q_k$.

The second term is

$$\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\frac{d\Phi_{p_l}^{r_k}}{\mathrm{d}t} = -r_k\left(\prod_j \frac{d^{z_j^k}\Phi_{p_l}}{d\sigma_l^{z_j^k}}\right)\frac{d\Phi_{p_l}}{d\sigma_l}\Phi_{p_l}^{r_k}.$$

This term has a product of $q_k + 1$ derivatives and, since we are evaluating for $m+1$, the following equation holds $(m+1) - r_k + 1 = m - r_k + 1 + 1 = q_k + 1$.

Hence, $\frac{d^{m+1}\Phi_{p_l}}{\mathrm{d}t^{m+1}}$ can be carried to the form (13), completing the proof. ∎

## ACKNOWLEDGEMENT

Esteban Jiménez acknowledges to CONACyT, México for the MSc scholarship number 426598.

## REFERENCES

[1] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15, no. 3, pp. 273–282, 1966.

[2] L. Weiss and E. Infante, "Finite time stability under perturbing forces and on product spaces," *IEEE Transactions on Automatic Control*, vol. 12, no. 1, pp. 54–59, February 1967.

[3] A. Michel and D. Porter, "Practical stability and finite-time stability of discontinuous systems," *IEEE Transactions on Circuit Theory*, vol. 19, no. 2, pp. 123–129, Mar 1972.

[4] V. Haimo, "Finite time controllers," *SIAM Journal on Control and Optimization*, vol. 24, no. 4, pp. 760–770, 1986.

[5] V. I. Utkin, *Sliding Modes in Control and Optimization*. Springer Verlag, 1992.

[6] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[7] E. Moulay and W. Perruquetti, “Lyapunov-based approach for finite time stability and stabilization,” in *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC ’05*, 2005, pp. 4742–4747.

[8] ——, “Finite-time stability and stabilization: State of the art,” in *Advances in Variable Structure and Sliding Mode Control*, ser. Lecture Notes in Control and Information Science, C. Edwards, E. Fossas Colet, and L. Fridman, Eds. Springer Berlin Heidelberg, 2006, vol. 334, pp. 23–41.

[9] V. Andrieu, L. Praly, and A. Astolfi, “Homogeneous approximation, recursive observer design, and output feedback,” *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[10] E. Cruz-Zavala, J. Moreno, and L. Fridman, “Uniform second-order sliding mode observer for mechanical systems,” in *Variable Structure Systems (VSS), 2010 11th International Workshop on*, june 2010, pp. 14 –19.

[11] A. Polyakov, “Nonlinear feedback design for fixed-time stabilization of linear control systems,” *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[12] L. Fraguela, M. Angulo, J. Moreno, and L. Fridman, “Design of a prescribed convergence time uniform robust exact observer in the presence of measurement noise,” in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, Dec 2012, pp. 6615–6620.

[13] A. Polyakov and L. Fridman, “Stability notions and Lyapunov functions for sliding mode control systems,” *Journal of the Franklin Institute*, vol. 351, no. 4, pp. 1831 – 1865, 2014, special Issue on 2010-2012 Advances in Variable Structure Systems and Sliding Mode Algorithms.

[14] E. Cruz-Zavala, J. A. Moreno, and L. M. Fridman, “Uniform robust exact differentiator,” *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2727–2733, Nov 2011.

[15] J. D. Sánchez-Torres, E. N. Sánchez, and A. G. Loukianov, “A discontinuous recurrent neural network with predefined time convergence for solution of linear programming,” in *IEEE Symposium on Swarm Intelligence (SIS)*, 2014, pp. 9–12.

[16] J. D. Sánchez-Torres, E. N. Sanchez, and A. G. Loukianov, “Predefined-time stability of dynamical systems with sliding modes,” in *American Control Conference (ACC), 2015*, July 2015, pp. 5842–5846.

[17] E. Jiménez-Rodríguez, J. D. Sánchez-Torres, D. Gómez-Gutiérrez, and A. G. Loukianov, “Predefined-time tracking of a class of mechanical systems,” in *13th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE), 2016*, 2016.