

On the least upper bound for the settling time of a class of fixed-time stable systems

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Abstract

This paper deals with the convergence time analysis of a class of fixed-time stable systems with the aim to provide a new non-conservative upper bound for its settling time. Our contribution is threefold. First, we revisit the well-known class of fixed-time stable systems, given in [1], while showing the conservatism of the classical upper estimate of the settling time. Second, we provide the smallest constant that uniformly upper bounds the settling time of any trajectory of the system under consideration. Then, introducing a slight modification of the previous class of fixed-time systems, we propose a new predefined-time convergent algorithm where the least upper bound of the settling time is set *a priori* as a parameter of the system. This calculation is a valuable contribution toward online differentiators, observers and controllers in applications with real-time constraints.

Keywords: Predefined-time stability, Finite-time stability, Fixed-time stability, Lyapunov analysis.

1. Introduction

Convergence time is an important performance specification for a controlled system from a practical point of view [2, 3]. Indeed, the design of controllers which guarantee predefined-time stability, instead of asymptotic stability, is one of the desired objectives, which appear in many applications such as missile guidance [4], hybrid formation flying [5], group consensus [6], online differentiators [7–9], state observers [2] etc. Furthermore, in the case of switching systems, it is frequently required that the observer (or controller) achieves the stability of the observation error (or tracking error) before the next switching [10, 11].

Lack of uniform boundedness of the settling time function regardless of the initial conditions causes several restrictions to the practical application of finite time observer/controller [12–14]. These restrictions can be relaxed using fixed-time stability concept. It is an extension of global finite-time stability, which guarantees the convergence (settling) time to be globally uniformly bounded, i.e., the bound does not depend on the initial state of the system [1, 4, 15]. To this end, the following class of systems, proposed in [1, 15],

$$\dot{x} = -(\alpha|x|^p + \beta|x|^q)^k \operatorname{sign}x, \quad x(0) = x_0, \quad (1)$$

where x is a scalar state variable, real numbers $\alpha, \beta, p, q, k > 0$ are system parameters which satisfy the constraints $kp < 1$ and $kq > 1$, have been extensively used. Indeed, it represents a wide class of systems which present the fixed-time stability property through homogeneity and Lyapunov analysis frameworks. However, it is still difficult to derive

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a relatively simple relationship between the system parameters and the upper bound of the settling time [16, 17]. It results in some difficulties in the tuning of the system parameters to achieve a prescribed-time stabilization (see for instance [7]).

The computation of the least upper bound of the settling time is usually not an easy task. Therefore, it is common to propose an upper bound of the settling time as an attempt to approximate the least one. For instance, in [1], it is shown that for system (1), the settling time $T(x_0)$ is bounded as

$$T(x_0) \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)} = T_{\max}, \quad \forall x_0 \in \mathbb{R}. \quad (2)$$

However, this bound significantly overestimates the least upper one. This overestimation can lead to restrictions for the practical implementation of prescribed-time observer/controller. In this case, the gains will be over-tuned to achieve a prescribed-time stabilization. It may lead to poor performances in terms of control magnitude or robustness against measurement noise for instance.

Considering the extensive use of the class of systems represented by (1) and the overestimation exhibited by T_{\max} in (2), this paper addresses the computation of the least upper bound of the settling time for this system and the derivation of a new predefined-time convergent algorithm where the least upper bound of the settling time is set *a priori* as a parameter of the system. In contrast to the common approaches of homogeneity and Lyapunov analysis, the results in this paper are derived using the well-known geometric conditions proposed in [12]. The overall contribution of this paper is divided into the following three main results:

1. We revisit the well-known class of fixed-time stable systems. Based on an appropriate use of the Gamma function, the smallest constant $\gamma(\rho)$, as a function of the parameter vector $\rho = [\alpha \beta p q k]^T \in \mathbb{R}^5$, that uniformly upper bounds the settling time of any trajectory of system (1) is provided.
2. The following new predefined-time convergent algorithm where the least upper bound of the settling time is set *a priori* as a parameter of the system is proposed:

$$\dot{x} = -\frac{\gamma(\rho)}{T_c}(\alpha|x|^p + \beta|x|^q)^k \text{sign}x, \quad x(0) = x_0, \quad (3)$$

where x is a scalar state variable, real numbers $\alpha, \beta, p, q, k > 0$ are system parameters which satisfy the constraints $kp < 1$ and $kq > 1$ and $T_c > 0$. Notice that the only difference between (1) and the modified system (3) is the constant gain γ/T_c . This slight change in the original system represents a considerable improvement in its properties, since the tunable parameter T_c is directly the least upper bound of the convergence time. As a consequence of this desirable feature, we say that the origin of system (3) is predefined-time stable with (strong) predefined-time T_c , a notion formally defined in Section 2.

3. The bound given in the formula (2) for system (3) is shown to be a conservative estimation of the settling time $T_f = \sup_{x_0 \in \mathbb{R}} T(x_0) = T_c$. Moreover, letting $\alpha = \varrho$ and $\beta = \frac{1}{\varrho}$, it is shown that even if the least upper bound of the convergence time is T_c , the upper estimate (2), given in [1] goes to infinity as $\varrho \rightarrow +\infty$ and as $\varrho \rightarrow 0$.

The rest of the manuscript is organized as follows. In Section 2, we introduce the preliminaries on finite-time, fixed-time and predefined-time stability. In Section 3, we present the main result on the least upper bound for the settling time and propose a new strongly predefined-time convergent algorithm where the least upper bound of the settling time is set *a priori* as a parameter of the system. In Section 4, we present the analysis of how conservative the bound provided in [1] may result and show some numerical results. Finally, in Section 5, we present some concluding remarks.

2. Preliminaries and Definitions

Consider the nonlinear system

$$\dot{x} = f(x; \rho), \quad x(0) = x_0, \quad (4)$$

where $x \in \mathbb{R}^n$ is the system state, the vector $\rho \in \mathbb{R}^b$ stands for the system (4) parameters which are assumed to be constant, i.e., $\dot{\rho} = 0$. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be nonlinear and continuous, and the origin is assumed to be an equilibrium point of system (4), so $f(0; \rho) = 0$.

Let us first recall some useful definitions and lemma on finite-time, fixed-time and predefined-time stability.

Definition 1. (Lyapunov stability [18, Definition 4.1]) *The origin of system (4) is said to be globally Lyapunov stable if for any $x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$, there is $\delta := \delta(\epsilon) > 0$ such that for all $x_0 \in \mathbb{R}^n$, if $\|x_0\| < \delta$, then $\|x(t, x_0)\| < \epsilon$ for all $t \geq 0$.*

Definition 2. (Finite-time stability [13]) *The origin of (4) is said to be globally finite-time-stable if it is globally Lyapunov stable, and for any $x_0 \in \mathbb{R}^n$, there exists $0 \leq T < +\infty$ such that the solution $x(t, x_0) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T : x(t, x_0) = 0, \forall t \geq T\}$ is called the settling time function.*

Lemma 1. (Finite-time stability characterization for scalar systems [12, Fact 1]) *Let $n = 1$ in system (4) (scalar system). The origin is globally finite-time stable if and only if for all $x \in \mathbb{R} \setminus \{0\}$*

(i) $xf(x; \rho) < 0$, and

(ii) $\int_x^0 \frac{dz}{f(z; \rho)} < +\infty$.

Remark 1. *A proof of Lemma 1 shall not be given here, but can be found in [19, Lemma 3.1]. Nevertheless, intuitively, condition (i) implies global Lyapunov stability. Moreover, under the conditions of Lemma 1, note that the settling time function is $T(x_0) = \int_0^{T(x_0)} dt$. Since first-order systems do not oscillate, the solution $x(\cdot, x_0) : [0, T(x_0)) \rightarrow [x_0, 0)$ of system (4) as a function of t defines a bijection. Using it as a variable change, the above integral equals (note that $\frac{1}{f(x; \rho)}$ is defined for all $x \in \mathbb{R} \setminus \{0\}$ from condition (i))*

$$T(x_0) = \int_0^{T(x_0)} dt = \int_{x_0}^0 \frac{dx}{f(x; \rho)}. \quad (5)$$

Thus, condition (ii) of Lemma 1 refers to the settling time function being finite.

Definition 3. (Fixed-time stability [1]) *The origin is said to be a fixed-time-stable equilibrium of (4) if it is globally finite-time-stable and the settling time function $T(x_0)$ is bounded on \mathbb{R}^n , i.e. $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.*

Remark 2. *Notice that there are multiple upper bounds of the settling time function T_{\max} ; for instance, if $T(x_0) \leq T_{\max}$, also $T(x_0) \leq \lambda T_{\max}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling time function.*

Definition 4. [20] *Let the origin be fixed-time-stable for system (4). The least upper bound of the settling time function, denoted by T_f , is the positive real number defined as*

$$T_f := \min_{T_{\max}} \{T_{\max} : T(x_0) \leq T_{\max}, \forall x_0 \in \mathbb{R}^n\} = \sup_{x_0 \in \mathbb{R}^n} T(x_0).$$

Remark 3. *It has been shown that fixed-time stability is guaranteed if the vector field of the system is homogeneous in the bi-limit, a concept defined in [15, 21]. However, in these cases, the upper bound for the settling time is usually not obtained. To differentiate this case to one where a settling time bound T_c is set in advance as a function of system parameters ρ , i.e. $T_c = T_c(\rho)$, we introduce the concept of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, i.e., T_c is the least upper bound for the settling time.*

Definition 5 (Predefined-time stability [22]). *For the parameter vector ρ of system (4) and a constant $T_c := T_c(\rho) > 0$, the origin of (4) is said to be predefined-time stable if it is fixed-time stable and the settling time function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ is such that $T(x_0) \leq T_c, \forall x_0 \in \mathbb{R}^n$. In this case, T_c is a predefined time.*

Moreover, if the settling time function is such that $T_f = T_c$, T_c is called the strong predefined time.

3. Main Result

Let us first revisit the well-known class of fixed-time stable systems. In the following theorem, based on an appropriate use of the Gamma function, the smallest constant that uniformly upper bounds the settling time of any trajectory of system (1) is provided.

Theorem 1. *Let ρ be the parameter vector $\rho = [\alpha \beta p q k]^T \in \mathbb{R}^5$ of (1) and let*

$$\gamma(\rho) = \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k\Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p}, \quad (6)$$

where $\Gamma(\cdot)$ is the Gamma function defined as $\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt$ [23, Chapter 1], and $m_p = \frac{1-kp}{q-p}$ and $m_q = \frac{kq-1}{q-p}$ are positive parameters.

The origin $x = 0$ of system (1) is fixed-time stable and the settling time function satisfies $T_f = \gamma$, where γ is defined in (6).

Proof. Note that for system (1), the field is $f(x; \rho) = -(\alpha|x|^p + \beta|x|^q)^k \text{sign}x$, where the parameter vector is $\rho = [\alpha \beta p q k]^T \in \mathbb{R}^5$. Furthermore, the product $xf(x; \rho) = -(\alpha|x|^p + \beta|x|^q)^k |x| < 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Thus, $V(x) = \frac{1}{2}x^2$ is a radially unbounded Lyapunov function for system (1), so its origin $x = 0$ is globally Lyapunov stable [18, Theorem 4.2].

Now, let $x_0 \in \mathbb{R} \setminus \{0\}$ (if $x_0 = 0$, then $x(t, 0) = 0$ is the unique solution of (1) and $T(0) = 0$). From (5), the settling time function is

$$\begin{aligned} T(x_0) &= \int_{x_0}^0 \frac{dx}{f(x; \rho)} \\ &= \int_0^{x_0} \frac{\text{sign}x dx}{(\alpha|x|^p + \beta|x|^q)^k}, \quad z = |x| \\ &= \int_0^{|x_0|} \frac{dz}{(\alpha z^p + \beta z^q)^k}. \end{aligned}$$

Since the integrand $\frac{1}{(\alpha z^p + \beta z^q)^k}$ is positive for $z \in (0, |x_0|)$, the settling time function is increasing with respect to $|x_0|$. Hence, the least upper bound of $T(x_0)$ (in the extended real numbers set, since we do not know if it is bounded or finite) is obtained using Proposition 1, in the Appendix, as

$$\sup_{x_0 \in \mathbb{R}} T(x_0) = \lim_{|x_0| \rightarrow +\infty} T(x_0) = \int_0^{+\infty} \frac{dz}{(\alpha z^p + \beta z^q)^k} = \gamma(\rho),$$

with $\gamma < +\infty$ as in (6). Using Lemma 1 and by Definitions 3 and 4, the origin $x = 0$ of system (1) is fixed-time stable and $T_f = \gamma$, which completes the proof. \square

Let us now study system (3). This new system consists in a slight change in the original system (1), e.g. constant gain γ/T_c is introduced. In the next theorem, we will show its interesting stability property.

Theorem 2. *The origin of (3) is predefined-time stable and T_c is the strong predefined-time.*

Proof. System (3) can be rewritten as:

$$\dot{x} = -(\hat{\alpha}|x|^p + \hat{\beta}|x|^q)^k \text{sign}x$$

where $\hat{\alpha} = \alpha \left(\frac{\gamma(\rho)}{T_c}\right)^{\frac{1}{k}}$ and $\hat{\beta} = \beta \left(\frac{\gamma(\rho)}{T_c}\right)^{\frac{1}{k}}$. Hence, using Theorem 1, the origin $x = 0$ of system (3) is fixed-time stable and the least upper bound of the settling time $T(x_0)$ is

$$\begin{aligned} T_f &= \frac{\Gamma(m_p)\Gamma(m_q)}{\hat{\alpha}^k\Gamma(k)(q-p)} \left(\frac{\hat{\alpha}}{\hat{\beta}}\right)^{m_p} \\ &= \frac{T_c}{\gamma(\rho)} \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k\Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p} = T_c \end{aligned}$$

which is directly a tunable parameter of the system. By Definition 5, the origin of (3) is in fact predefined-time stable and T_c is the strong predefined-time. \square

Remark 4. In [24], the least upper estimation of the settling time of (1) was addressed for the case where $k = 1$, $p = 1 - s$, $q = 1 + s$, with $0 < s < 1$, where $\gamma(\rho)$ reduces to $\gamma(\rho) = \frac{\Gamma(\frac{1}{2})^2}{2s\sqrt{\alpha\beta}} = \frac{\pi}{2s\sqrt{\alpha\beta}}$. In [25], it was shown that with $\alpha = \beta = \frac{\pi}{2sT_c}$ the least upper estimation of the settling time is T_c . Thus, Theorem 1 and Theorem 2 are a generalization of the results presented in [24] and [25], respectively. Since, only for the case where $k = 1$, $p = 1 - s$, $q = 1 + s$, with $0 < s < 1$, there exist results in the literature with a non-conservative upper bound estimate, many applications, for instance fixed-time consensus protocols [26–28], have been restricted to this case.

With Theorem 2, the Lyapunov characterization of fixed time stability presented in [1, Lemma 1] can be rewritten to obtain predefined-time stability.

Theorem 3. (A Lyapunov characterization for predefined-time systems) Assume there exist a continuous radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} V(0) &= 0, \\ V(x) &> 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \end{aligned}$$

and the derivative of V along the trajectories of (4) satisfy

$$\dot{V}(x) \leq -\frac{\gamma(\rho)}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (7)$$

where $\alpha, \beta, p, q, k > 0$, $kp < 1$, $kq > 1$ and γ is given in (6).

Then, the origin of (4) is predefined-time-stable and T_c is a predefined time. If in addition, the equality holds in (7), then T_c is the strong predefined-time.

Proof. Suppose that there exist a function $w(t) \geq 0$ that satisfies

$$\dot{w} = -\frac{\gamma(\rho)}{T_c} (\alpha w^p + \beta w^q)^k$$

and $V(x_0) \leq w(0)$. Hence, by Theorem 2, $w(t)$ will converge to the origin in a strong predefined-time T_c . Moreover, by the comparison lemma [18, Lemma 3.4], it follows that $V(x(t)) \leq w(t)$, with equality only if (7) is an equality. Consequently, $V(x(t))$ will converge to the origin in a predefined time T_c . If (7) is an equality and $V(x_0) = w(0)$, then, $V(x(t)) = w(t)$ and $V(x(t))$ will converge in the strong predefined-time T_c . \square

Given the relevance of fixed-time stability argument in the introduction, the inequality (7) is an important consequence of the main results, since as we show next, the upper estimate (2) is often too conservative. Thus, applications based on the upper estimate of the settling time (2) presented in [1, Lemma 1] are often over-engineered.

4. Settling time bound analysis and comparison

Consider the system given in (3), which according to Theorem 2, is predefined-time stable with strong predefined-time T_c . However, it follows from (2) that an upper estimate for the settling time $T(x_0)$ is

$$T(x_0) \leq \frac{T_c}{\gamma(\rho)} \left(\frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)} \right), \quad \forall x_0 \in \mathbb{R}. \quad (8)$$

Let $\varrho > 0$, $\alpha = \varrho$ and $\beta = \frac{1}{\varrho}$. Assuming that p, q and k remain constant, and noticing that $\gamma(\rho)$ is a function of ϱ , it can be seen that varying ϱ the least upper bound of the settling time remains $T_f = T_c$. However, the bound (8) becomes

$$T(x_0) \leq T_{\max}(\varrho) := \frac{T_c}{K} \left(\frac{1}{\varrho^{2m_p}(1-pk)} + \frac{\varrho^{2(k-2m_p)}}{(qk-1)} \right), \quad (9)$$

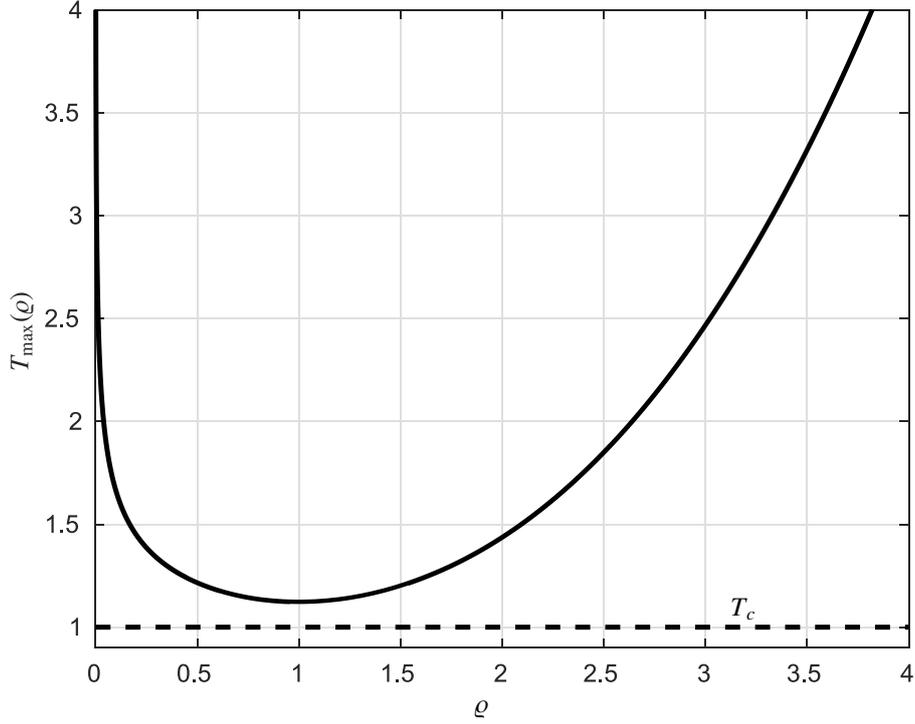


Figure 1: Function $T_{\max}(\varrho)$ for $p = 0.5$, $q = 3$, $k = 1.5$. The least upper bound of the settling time is $T_c = 1s$. The smallest value of $T_{\max}(\varrho)$ is $\min_{\varrho>0} T_{\max}(\varrho) = T_{\max}(1) = 1.1249s$.

where $K = \frac{\Gamma(m_p)\Gamma(m_q)}{\Gamma(k)(q-p)}$. It is easy to see that

$$\lim_{\varrho \rightarrow 0} T_{\max}(\varrho) = +\infty \quad \text{and} \quad \lim_{\varrho \rightarrow \infty} T_{\max}(\varrho) = +\infty,$$

i.e. $T_{\max}(\varrho)$ in (9) has no upper bound as ϱ increases or it is close to zero.

Moreover, the best upper estimate of the bound (9) is achieved at $\arg \min_{\varrho>0} T_{\max}(\varrho) = 1$, and its value is

$$\min_{\varrho>0} T_{\max}(\varrho) = T_{\max}(1) = \frac{T_c}{K} \left(\frac{1}{(1-pk)} + \frac{1}{(qk-1)} \right) > T_c.$$

An illustration of this argument, showing $T_{\max}(\varrho)$ as a function of ϱ , with $T_c = 1s$, is presented in Figure 1. Although, by Theorem 2 the least upper bound of the settling time is $T_c = 1$, it can be seen that in the best case the bound (8) provides an overestimation of εT_c with $\varepsilon = \frac{1}{K} \left(\frac{1}{(1-pk)} + \frac{1}{(qk-1)} \right) \Big|_{p=0.5, q=3, k=1.5} = 1.1249$.

On the other hand, a numerical simulation of system (3), with the parameters $\varrho = 4$, $p = 0.5$, $q = 3$, $k = 1.5$, for several initial conditions x_0 , is presented in Figure 2. Again, it can be seen that the least upper bound of the settling time is $T_c = 1s$ but the bound (8) for these parameter values is $T_{\max}(4) = 4.4331s$.

5. Conclusions and future work

In this paper, we revisited the fixed-time stability problem for a class of nonlinear systems. We showed that the well-known upper bound condition for the settling time of this class of systems is often too conservative. To illustrate our claim, we showed how by changing one parameter the upper estimate of the settling time tends to infinity even though the actual settling time is always bounded by a constant T_c . To address this problem, we proposed a modification to the classical fixed-time algorithm to transform it into a strongly predefined-time (in which the upper bound for the settling time is set in advance as a parameter of the system and is the lowest upper estimate of the settling

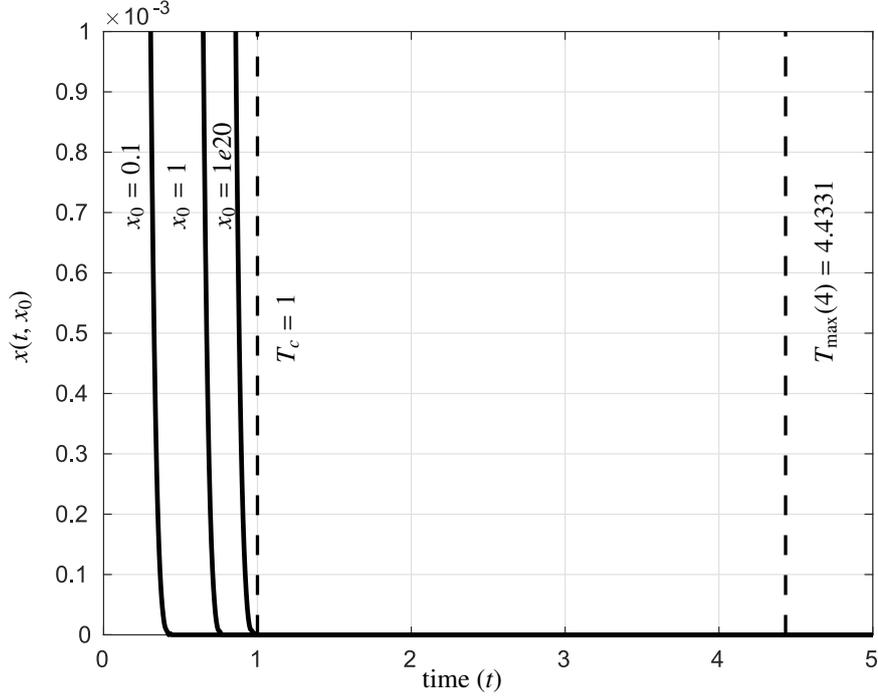


Figure 2: Trajectories of (3) for different initial conditions with $T_c = 1s$, $\varrho = 4$, $p = 0.5$, $q = 3$ and $k = 1.5$. The upper estimate in (9) is $T_{\max}(4) = 4.4331s$. The least upper estimate is $T_c = 1s$

time) with strong predefined-time T_c . With this result, the Lyapunov inequality, which is a sufficient condition for fixed-time stability, was modified in a way that becomes predefined-time parametrized by T_c . When such inequality becomes equality, T_c becomes the lowest upper estimate of the settling time of the system. This computation is an essential contribution toward online differentiators and observers and controllers satisfying prescribed-time objectives.

Appendix A. Auxiliary Results

Definition 6. [23, Pg. 87] Let $a, b > 0$. The Beta function, denoted by $B(a, b)$, is defined as

$$B(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proposition 1. Let $\alpha, \beta, p, q, k > 0$, with $pk < 1$, $qk > 1$. Hence,

$$\int_0^{+\infty} \frac{dx}{(\alpha x^p + \beta x^q)^k} = \frac{\Gamma(m_p)\Gamma(m_q)\left(\frac{\alpha}{\beta}\right)^{m_p}}{\alpha^k \Gamma(k)(q-p)}. \quad (\text{A.1})$$

Proof. The left-hand side of (A.1) can be rewritten as

$$\int_0^{+\infty} \frac{dx}{(\alpha x^p + \beta x^q)^k} = \int_0^{+\infty} \frac{\beta^k x^{qk} dx}{\left(\frac{\alpha}{\beta} x^{p-q} + 1\right)^k}. \quad (\text{A.2})$$

Note that the term $z = \left(\frac{\alpha}{\beta} x^{p-q} + 1\right)^{-1}$ goes to 0 when $x \rightarrow 0$ and to 1 when $x \rightarrow 1$ if $p - q < 0$. Furthermore, using z

as a variable change and by Definition 6, the integral (A.2) becomes

$$\begin{aligned} \int_0^{+\infty} \frac{dx}{(\alpha x^p + \beta x^q)^k} &= \frac{\left(\frac{\alpha}{\beta}\right)^{m_p}}{\alpha^k(q-p)} \int_0^1 z^{m_p-1}(1-z)^{m_q-1} dz \\ &= \frac{B(m_p, m_q)}{\alpha^k(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p} \\ &= \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k\Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p}, \end{aligned}$$

concluding the proof. □

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