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Liquidity-Adjusted Sharpe Ratio

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Sean Nicolás González Vázquez

Abstract

Modern Portfolio Theory, introduced by Harry Markowitz in 1952, and the Sharpe Ratio, proposed by William Sharpe in 1966, focus on portfolio optimization considering market return and risk. Markowitz and Sharpe developed these theories in the New York Stock Exchange (NYSE) context. However, in markets with different characteristics, such as the Mexican market, additional factors beyond returns and market risk play a crucial role in the performance of an investment portfolio. One of the major challenges in low-liquidity markets is the difficulty of quickly buying or selling assets due to liquidity constraints.

This thesis introduces a novel approach to portfolio optimization in such environments: the *Liquidity-Adjusted Sharpe Ratio*. This method seeks to maximize the risk-return profile of a portfolio while factoring in liquidity risk, using a liquidity shrinkage factor to penalize assets with low liquidity. By doing so, the model balances an asset's return potential with its associated market risk and liquidity risk. Additionally, we propose a specific case of this approach, termed the Liquidity Variance Return Ratio (LVRR), which measures the Sharpe Ratio achieved per unit of liquidity risk taken in an investment. The higher the LVRR, the more efficiently the portfolio manages liquidity risk in relation to its returns. The goal, therefore, is to maximize the LVRR, making it a distinct case within the broader Liquidity-Adjusted Sharpe Ratio framework. This targeted optimization ensures that portfolios are not only risk-efficient but also liquid, which is crucial in constrained markets.

Throughout this document, we present this method's analytical and empirical solution and propose a Sharpe-Liquidity efficient frontier that considers the return, market risk, and liquidity risk of an investment portfolio. This new frontier offers a more *ad hoc* approach for investors facing liquidity constraints when trading. We demonstrated that the proposed method and its elements (such as the LVRR portfolio) fall within this front's optimal and feasible region. Also, we found that to achieve lower liquidity risk in a portfolio, it is generally necessary to accept higher market risk. However, in most cases, and subject to the selection of portfolio assets, the returns of LVRR-optimized portfolios show lower kurtosis with higher positive skewness compared to optimal Maximum Sharpe portfolios. This means that although the volatility of a LVRR portfolio generally increases, this volatility is skewed towards gains, i.e., positive returns.

Additionally, when implementing the method through the backtesting of portfolios with random asset selection, we found that our method often outperforms the traditional Sharpe ratio approach in the Mexican market. Specifically, our analysis reveals a 74% probability of achieving superior returns in static long-term portfolios and a 80% probability in dynamic portfolios that incorporate periodic rebalancing. We also found that in liquid markets, the method tends to converge to the Sharpe Ratio, making it better to adopt the classic approach in these contexts due to the low relevance of liquidity risk. After extensive research, it has been concluded that the LVRR method consistently delivers favorable outcomes within low-liquidity markets. This suggests that the proposed methodology can serve as a viable alternative for investors seeking to operate effectively in such environments.

Ratio de Sharpe Ajustado por Liquidez

Sean Nicolás González Vázquez

Resumen

La Teoría Moderna de Portafolios, introducida por Harry Markowitz en 1952, y el Ratio de Sharpe, propuesto por William Sharpe en 1966, se centran en la optimización de portafolios considerando el rendimiento y el riesgo del mercado. Markowitz y Sharpe desarrollaron estas teorías en el contexto de la Bolsa de Valores de Nueva York (NYSE). Sin embargo, en mercados con características diferentes, como el mercado mexicano, factores adicionales más allá de los rendimientos y el riesgo de mercado juegan un papel crucial en el desempeño de un portafolio de inversión. Uno de los principales desafíos en mercados de baja liquidez es la dificultad de comprar o vender activos rápidamente debido a restricciones de liquidez.

Esta tesis introduce un enfoque novedoso para la optimización de portafolios en tales entornos: el *Ratio de Sharpe Ajustado por Liquidez*. Este método busca maximizar el perfil de riesgo-rendimiento de un portafolio al mismo tiempo que considera el riesgo de liquidez, usando un factor de regularización que penaliza los activos con baja bursatilidad. De este modo, el modelo equilibra el rendimiento con su riesgo de mercado y de liquidez asociado. Adicionalmente, proponemos un caso específico de este enfoque, denominado el Ratio de Retorno-Varianza-Liquidez (LVRR), que mide el Ratio de Sharpe obtenido por unidad de riesgo de liquidez asumido en una inversión. El objetivo, por lo tanto, es maximizar el LVRR, convirtiéndolo en un caso distintivo dentro del marco más amplio del Ratio de Sharpe Ajustado por Liquidez. Esta optimización asegura que los portafolios no solo sean eficientes en términos de riesgo de mercado y rendimiento, sino también líquidos.

A lo largo de este documento, presentamos la solución analítica y empírica de este método y proponemos una frontera eficiente en Sharpe-Liquidez que considera el retorno, el riesgo de mercado y el riesgo de liquidez de un portafolio de inversión. Esta nueva frontera ofrece un enfoque más *ad hoc* para los inversionistas que enfrentan restricciones de liquidez al operar. Demostramos que el método propuesto y sus elementos (como el portafolio LVRR) se encuentran dentro de la región óptima y factible de esta frontera. También encontramos que para lograr un menor riesgo de liquidez en un portafolio, generalmente es necesario aceptar un mayor riesgo de mercado. Sin embargo, en la mayoría de los casos, y sujeto a la selección de los activos del portafolio, los rendimientos de los portafolios optimizados con LVRR muestran menor curtosis y mayor asimetría positiva en comparación con los portafolios óptimos de Máximo Sharpe. Esto significa que, aunque la volatilidad de un portafolio LVRR generalmente aumenta, esta volatilidad está sesgada hacia las ganancias.

Además, al implementar el método usando backtesting de portafolios con selección aleatoria de activos, encontramos que nuestro método a menudo supera al enfoque tradicional del Ratio de Sharpe en el mercado mexicano. Específicamente, nuestro análisis revela una probabilidad del 74% de lograr rendimientos superiores en portafolios estáticos a largo plazo y del 80% en portafolios dinámicos que incorporan rebalancesos periódicos. También encontramos que, en mercados líquidos, el método tiende a converger al Ratio de Sharpe, lo que hace preferible adoptar el enfoque clásico en estos contextos debido a la baja relevancia del riesgo de liquidez. Tras una extensa investigación, se ha concluido que el método LVRR ofrece consistentemente resultados favorables en mercados de baja liquidez. Esto sugiere que la metodología propuesta puede servir como una alternativa viable para inversionistas que buscan operar de manera efectiva en dichos entornos.

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Dedicated to Guadalupe Vázquez

Dedicado a Guadalupe Vázquez

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Notation

w_i Weight of the i -eth asset within a portfolio.

w Vector of a set of assets weights.

μ_i Expected return of the i -eth asset within a portfolio.

μ Vector of expected returns of a set of assets.

μ_p Expected return of the portfolio.

Σ Covariance matrix of asset returns.

Σ^{-1} Inverse covariance matrix of asset returns.

\in Set membership symbol that means "belongs to" or "is an element of."

\mathbb{R} Set of all real numbers.

σ_p^2 Portfolio variance.

σ_p Portfolio volatility.

$\lambda, \lambda_1, \lambda_2$ Lagrange multipliers for the optimization constraints.

1 Scalar representing the value 1.

$\mathbb{1}$ Vector of ones.

∇L Gradient operator of the Lagrangian function.

U Matrix containing μ and $\mathbb{1}$.

u Matrix containing μ_p and 1.

M Matrix resulting from $U^T \Sigma^{-1} U$.

s_{11} Scalar: $\mathbb{1}^T \Sigma^{-1} \mathbb{1}$.

$s_{1\mu}$ Scalar: $\mathbb{1}^T \Sigma^{-1} \mu$.

$s_{\mu\mu}$ Scalar: $\mu^T \Sigma^{-1} \mu$.

r_f Risk-free rate.

w_{rf} Weight of the risk-free asset within a portfolio.

SR Sharpe Ratio.

∇f Gradient of function f .

σ_u Upside risk.

σ_d Downside risk.

$F(x)$ Cumulative density function.

Ω Omega Ratio.

S Sortino Ratio.

x Decision variables in an optimization problem.

$\|Ax - b\|^2$ L2 norm.

x^{po} Pareto optimal point.

$f(x; p)$ Objective function in multi-objective optimization.

$g_j(x; p)$ Inequality constraints in multi-objective optimization.

$h_l(x; p)$ Equality constraints in multi-objective optimization.

N_{days} Number of days required to fully liquidate a position in a financial asset.

N_{shares} Number of shares.

ADV Average daily volume.

ADV_i Average daily volume of the i -eth asset within a portfolio.

C Capital or total value of a portfolio.

P Spot price of a financial asset.

P_i Spot price of the i -eth asset within a portfolio.

λ Liquidity shrinkage factor.

H Vector consisting of the reciprocals of the product of each asset's average daily volume and its spot price for all assets in the portfolio.

$LA SR$ Liquidity-Adjusted Sharpe Ratio.

$LVRR$ Liquidity Variance Return Ratio.

1 Introduction

Portfolio Managers, defined as “professionals who are responsible for making decisions and implementing investment strategies that meet the objectives and constraints of a client, construct and manage portfolios, make decisions on what and when to buy and sell investments”¹, face the challenge of allocating available capital among various financial assets.

This asset allocation challenge, colloquially known as the “portfolio problem”, was first quantitatively addressed in 1952 by Harry Markowitz² when he introduced two of the most important contributions to the stock market theory: minimum variance optimization and the mean-variance efficient frontier.

Years later, in 1966, William F. Sharpe proposed the Sharpe ratio maximization method³, which considered not only the variance for portfolio optimization but also the expected return.

Although investors widely use these two methods, one of their main drawbacks is that they only consider market risk despite the importance of other types of risks.

The main types of risk in the stock market are:

- **Market Risk:** Risk of incurring losses in a position due to fluctuations in the price of an asset⁴.
- **Market Liquidity Risk:** Risk of incurring a loss when a market participant tries to liquidate a position and does not get the best available price⁵.
- **Credit Risk:** Risk that a bank borrower or counterparty will fail to meet their obligations in an agreement⁶.
- **Operational Risk:** Risk of loss resulting from inadequate or failed internal processes, people or systems⁷.

¹ CFA. Portfolio manager, 2024. URL <https://www.cfainstitute.org/en/programs/cfa/charterholder-careers/roles/portfolio-manager>

² Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

³ William F. Sharpe. *Portfolio Theory and Capital Markets*. McGraw-Hill, 1966

⁴ Alexander McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton University, 2005

⁵ Basel Committee. Principles for sound liquidity risk management and supervision. *Bank for International Settlements Basel*, 2008

⁶ Basel Committee. Principles for the management of credit risk. *Bank for International Settlements Basel*, 1999

⁷ Basel Committee. International convergence of capital measurement and capital standards. *Bank for International Settlements Basel*, 2006

In markets with high stock concentration, such as the U.S. market⁸, the liquidity risk of most stocks is low. However, in other markets like the Mexican market, this factor is paramount when investing, as there is a large number of low-liquidity assets.

In these markets, liquidity risk becomes significantly relevant. The previously mentioned methods, developed in high-liquidity environments, may be subject to various biases by not considering this type of risk, potentially recommending investments in assets with limited market depth or sporadic trading.

Throughout history, numerous studies have been conducted to explain and quantify the liquidity risk of a financial asset⁹. Additionally, international regulators have established guidelines for the proper measurement and standardization of liquidity risk in the stock market context¹⁰, and textbooks dedicate entire chapters to the qualitative selection of assets based on their trading volume¹¹.

Furthermore, some research efforts have attempted to incorporate liquidity risk into the measurement of market risk in an investment portfolio¹², based on stochastic methodologies¹³. However, to date, there has yet to be a quantitative model for asset allocation in portfolios that comprehensively considers liquidity risk.

Therefore, the central purpose of this thesis is to present and develop an asset allocation method that improves the Sharpe ratio in low-liquidity markets, focusing on the Mexican market. In this new approach, we aim to take into account expected return, market risk, and the liquidity risk associated with financial assets¹⁴ for portfolio optimization.

To achieve this, we will take the Sharpe function and add a liquidity regularization term to maximize the return-to-risk ratio in an investment portfolio while penalizing assets with lower liquidity.

Under this methodology, we also seek to propose a new efficient frontier of portfolios that considers these three factors. Additionally, we will conduct portfolio backtestings with random asset selection to evaluate the practical effectiveness of our method.

⁸ The top ten U.S. stocks account for 29.4% of global market capitalization.

JP Morgan Equity Strategy. Stock market concentration it's on the rise, 2024. URL <https://www.jpmorgan.com/insights/global-research/markets/market-concentration>

⁹ Thierry Roncalli and Guillaume Weisang. Asset management and systemic risk. *Secretariat of Financial Stability Board*, 2015

¹⁰ Kleopatra Nikolaou. Liquidity concepts. *European Central Bank*, 2009

¹¹ Philippe Jorion. *Financial Risk Manager Handbook*. Wiley Finance, 2011

¹² Yu Tian. Market liquidity risk and market risk measurement. Master's thesis, Delft University of Technology, 2009

¹³ Carlo Acerbi and Giacomo Scandolo. Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, Vol. 8, pp. 681-692, 2008

¹⁴ Liquidity risk is systematic and non-diversifiable

Kleopatra Nikolaou. Liquidity concepts. *European Central Bank*, 2009

2 Preliminaries

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2.1 Investment Portfolios

An investment portfolio is a collection of assets, such as stocks, bonds, and other financial instruments, in which an investor allocates their capital. The practice of investing in a variety of assets to reduce total market risk is called diversification, and it is a fundamental pillar of portfolio theory. This is based on the premise that by including different types of assets in a portfolio, one can mitigate market risk since these assets behave differently according to their intrinsic characteristics¹.

Diversification is based on the correlation between assets. Some assets may have a negative correlation, meaning that when one increases

¹ Edwin Elton, Martin Gruber, Stephen Brown, and William Goetzmann. *Modern Portfolio Theory and Investment Analysis*. John Wiley and Sons, 2003

in value, the other may decrease. On the other hand, some assets may have a positive correlation, where both increase or decrease in value simultaneously. In summary, correlation measures the linear relationship between the returns of different investments.

Additionally, there is the possibility that two assets correlate close to zero, indicating that there is no significant linear relationship between them. For a portfolio to achieve adequate diversification, the recommendation is to include assets that are uncorrelated with each other, meaning a correlation close to zero, which provides greater stability and risk reduction to the portfolio².

² Mosek. *Portfolio Optimization Cookbook*. Mosek ApS, 2024

By diversifying, the goal is to minimize the negative impact that any individual asset could have on the overall portfolio performance. Therefore, by having a combination of assets with different correlations, one can reduce the idiosyncratic portfolio risk and potentially improve risk-adjusted returns. This strategy is essential for efficiently managing investments and achieving an investor's financial objectives.

In addition, portfolio optimization is a fundamental process in investment management that involves finding the "best" weightings for a set of assets for a specific purpose. The main objective of this process is to decide how much capital to allocate in the available assets, which is colloquially known as the "portfolio problem".

This quantitative asset allocation process is a rational investment method based on the use of mathematical models and statistical techniques to determine the "best" combination of investments based on a metric such as the relationship between assets. The quantitative methodology allows for more precise and objectively grounded decision-making, facilitating the identification of investment opportunities and risk management.

In this chapter, fundamental concepts related to portfolio construction and the methodologies and tools used for its optimization will be addressed. Additionally, concepts related to liquidity risk and other fundamentals necessary for the construction of this work will be explored, providing a solid foundation for understanding the theory and practice of portfolio optimization in markets with specific characteristics.

2.1.1 Portfolio Optimization

The process of investing in a set of assets—an investment portfolio—can be divided into two stages. The first stage consists of observation and experience and ends with the selection of the available stocks. The second stage begins with beliefs about the future and concludes with the allocation of these stocks in an investment portfolio³. In this thesis, we will focus on the second stage, assuming that investors are rational and seek to maximize the risk-return ratio.

To find the portfolio that maximizes the risk-return ratio, it is essential to define and understand several key concepts related to the *Modern Portfolio Theory* proposed by Harry Markowitz and complemented by William Sharpe. This theory provides a mathematical framework for constructing investment portfolios that minimize market risk or maximize expected return for a given level of market risk. Before delving into these concepts, it is crucial to understand how to measure these two factors. To measure the expected return of a portfolio with n assets, we calculate the weighted sum of the daily returns of the constituent assets, adjusted by their respective weights.

$$\mu_p = \sum_{i=1}^n w_i \mu_i \quad (2.1)$$

These daily returns are known a priori by a portfolio manager and exhibit interesting statistical properties⁴, such as their normal distribution with mean $\mu = 0$ and constant variance $\sigma^2 = c$. This information allows the manager to calculate the expected return of an investment portfolio, representing the long-term expected gain from the investment. On the other hand, the weights represent the proportion of capital allocated to each asset within the portfolio and are determined by the manager. These weights sum to one and are non-negative, as we will assume that only long positions are allowed in the portfolios. Consequently, short selling is prohibited⁵.

Rewriting in matrix form:

$$\mu_p = w^T \mu \quad (2.2)$$

Where:

- $w \in \mathbb{R}^n$: Weight vector
- $\mu \in \mathbb{R}^n$: Return vector

Another important factor in portfolio construction is the variance, which represents the market risk of the investment. Variance is

³ Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

⁴ Ludwig Chincarini and Daehwan Kim. *Quantitative Equity Portfolio Management*. McGraw Hill, 2006

⁵ A long position refers to buying an asset with the expectation that its value will increase, whereas short selling involves selling an asset that the investor does not own, anticipating a decline in its price to repurchase it later at a lower cost, thereby profiting from the difference.

Benjamin Graham. *The Intelligent Investor*. Harper Collins, 1949

calculated using the covariance matrix of the stock's daily returns and the asset weights. The covariance matrix indicates how two or more assets fluctuate between each other, measuring the joint direction of their movements. This matrix, denoted as Σ , is a square, symmetric, and invertible matrix.

The interpretation of variance is analogous to that of correlation, which also measures the relationship between the movements of n assets but on a standardized scale between -1 and 1. While the covariance matrix provides an absolute measure of how assets vary in their original units, the correlation matrix standardizes this measure, eliminating the units and allowing for a direct comparison of the relationships between assets regardless of their magnitudes⁶. To calculate the variance and the standard deviation of an investment portfolio, the following formulas are used:

$$\sigma_p^2 = w^T \Sigma w \quad (2.3)$$

$$\sigma_p = \sqrt{w^T \Sigma w} \quad (2.4)$$

Where:

- $w \in \mathbb{R}^n$: Weight vector
- $\Sigma \in \mathbb{R}^{n \times n}$: Covariance matrix

Variance in a portfolio represents the dispersion of the portfolio's returns relative to its mean, indicating how much the returns can vary compared to this average. The greater the variance, the higher the uncertainty, making this metric a measure of market risk.

With this metric, we can calculate the standard deviation, also known as volatility, which is the square root of the variance and is used to measure the same dispersion but in units of return. This makes it more interpretable, as it is on the same scale as the portfolio's returns. The standard deviation reflects the typical magnitude of the fluctuations of returns around their mean, providing a clear idea of the inherent risk of the portfolio.

Connecting with what was mentioned earlier, given that the daily returns of assets are normally distributed, it is essential to note that to calculate the expected performance and risk metrics of a portfolio composed of these assets, we use the primary measures of central tendency (mean) and dispersion (variance and standard deviation) of a normal distribution.

⁶ Mosek. *Portfolio Optimization Cookbook*. Mosek ApS, 2024

2.1.2 Minimum Variance Efficient Portfolio

The minimum variance portfolio⁷ is the portfolio that provides the lowest possible market risk given a set of n assets with weights w_1, w_2, \dots, w_n .

The central idea of this method is that it is possible to reduce the total risk of the portfolio by leveraging on the statistical properties of asset returns, specifically their covariances and variances. Markowitz demonstrated that by considering not only the expected returns of the assets but also how their returns move together (i.e., their covariance), investors can construct a portfolio that minimizes market risk.

This approach revolutionized investment management by presenting a quantitative framework that allows portfolio managers to allocate their capital in a more informed and data-driven manner. The optimization problem is defined as:

$$\min \sigma^2 : w^T \Sigma w \quad (2.5)$$

$$\text{subject to } w^T \mathbf{1} = 1$$

Where $\mathbf{1} \in \mathbb{R}^n$ and it represents a vector of ones, and the scalar 1 denotes the sum constraint on the weights. Note that the constraint indicates that the sum of the weights must equal one, which implies that we cannot invest beyond our available capital. We can solve this problem analytically using Lagrangian optimization⁸:

$$L = w^T \Sigma w + \lambda(w^T \mathbf{1} - 1)$$

$$\nabla L = \begin{bmatrix} 2\Sigma w + \lambda \mathbf{1} \\ w^T \mathbf{1} - 1 \end{bmatrix}$$

We set the first equation of the gradient to 0 and solve for w :

$$2\Sigma w + \lambda \mathbf{1} = 0$$

$$2\Sigma w = -\lambda \mathbf{1}$$

$$w = -\frac{1}{2} \lambda \Sigma^{-1} \mathbf{1}$$

We substitute w into the second gradient equation to find λ :

$$\left(-\frac{1}{2} \lambda \Sigma^{-1} \mathbf{1}\right)^T \mathbf{1} - 1 = 0$$

⁷ Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

⁸ Robert Merton. An analytic derivation of the efficient portfolio frontier. *Cambridge University Press*, Vol. 7, pp. 1851-1872, 1972

$$\lambda = -\frac{2}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

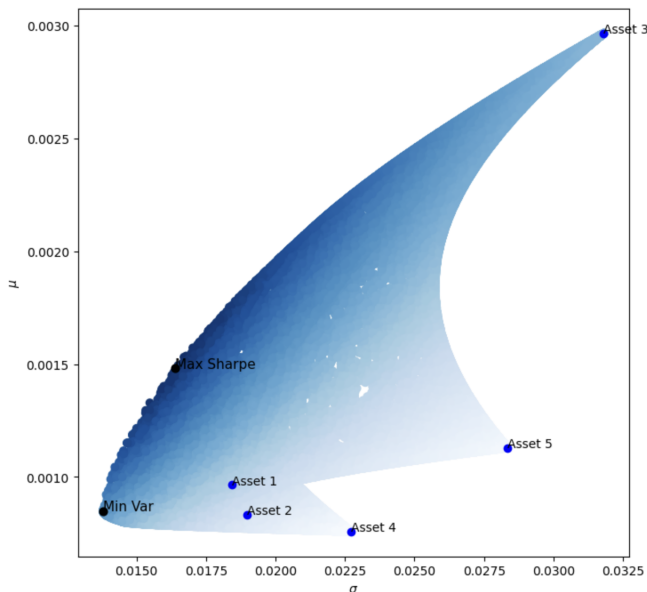
We substitute λ into the equation for w and find the optimal minimum variance weights:

$$\begin{aligned} w &= -\frac{1}{2} \lambda \Sigma^{-1} \mathbf{1} \\ w &= -\frac{1}{2} \left(-\frac{2}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \right) \Sigma^{-1} \mathbf{1} \\ w &= \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \end{aligned} \quad (2.6)$$

This optimization seeks the combination of assets that minimizes the total risk of the portfolio, making it suitable for investors with high-risk aversion⁹.

2.1.3 Mean-Variance Efficient Frontier

The Mean-Variance efficient frontier is a critical concept in modern portfolio theory and was proposed by Markowitz¹⁰. To explain this concept, we first plot the expected return (y-axis) and standard deviation (x-axis) for all possible combinations of assets in a given investment portfolio:



Here, we can observe that there are a finite number of combinations for the given five assets, where each portfolio has a certain volatility

⁹Risk aversion is defined as the preference for less volatile alternatives, keeping other variables such as expected return constant.

D. L. Hanson and C.F. Menezes. Risk aversion and bidding theory. *University of Kansas Press*, 1968

¹⁰ Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

Figure 2.1: Mean-Variance frontier with all possible combinations for a portfolio with five assets

and return. Note that for the portfolios on the upper margin of the frontier, starting from the minimum variance portfolio and ending at asset 2, there is no portfolio with a higher return for its level of risk. This set is defined as the *mean-variance efficient frontier*¹¹.

Robert Merton defines the efficient frontier as the *feasible set of portfolios that have the highest expected return for a given level of risk*¹². Assuming that investors are rational, they will always prefer higher returns for higher levels of risk. Therefore, for any set of assets, the portfolios within the efficient frontier are "*optimal*". Merton also analytically developed this concept by modifying the optimization problem presented in section 2.1.2, but considering the expected return of the portfolio in the constraints.

$$\begin{aligned} \min \sigma^2 : & \quad w^T \Sigma w & (2.7) \\ \text{subject to } & w^T \mathbf{1} = 1 \\ & w^T \mu = \mu_p \end{aligned}$$

The last constraint was added to the problem to consider the expected return of the portfolio. Here, $\mu \in \mathbb{R}^n$ is a vector that contains the individual asset returns, and μ_p is the scalar that represents the portfolio return. Hence, this return will equal the weighted sum of the individual assets returns.

To develop the efficient frontier concept analytically, we first optimize the weights for the portfolios within the frontier, allowing their calculation as a function of return and risk. Second, we demonstrate that the portfolio frontier is a hyperbola. Finally, we introduce the concept of the Capital Allocation Line (CAL), which is the tangent line to the frontier.

1.- Obtaining Weights for Portfolios within the Frontier

We obtain the Lagrangian of the optimization problem posed earlier, along with its gradient:

$$\begin{aligned} L &= w^T \Sigma w + \lambda_1 (w^T \mu - \mu_p) + \lambda_2 (w^T \mathbf{1} - 1) \\ \nabla L &= \begin{bmatrix} 2\Sigma w + \lambda_1 \mu + \lambda_2 \mathbf{1} \\ w^T \mu - \mu_p \\ w^T \mathbf{1} - 1 \end{bmatrix} \end{aligned}$$

We set the first equation to zero and solve for w :

¹¹ Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

¹² Robert Merton. An analytic derivation of the efficient portfolio frontier. *Cambridge University Press*, Vol. 7, pp. 1851-1872, 1972

$$2\Sigma w + \lambda_1 \mu + \lambda_2 \mathbf{1} = 0$$

$$w = -\frac{1}{2}\Sigma^{-1}\lambda_1\mu + -\frac{1}{2}\Sigma^{-1}\lambda_2\mathbf{1}$$

To find the values of λ_1 and λ_2 , we substitute w into the second and third equations of the gradient and set them to zero. For the second equation:

$$w^T \mu - \mu_p = 0$$

$$\left(-\frac{1}{2}\Sigma^{-1}\lambda_1\mu + -\frac{1}{2}\Sigma^{-1}\lambda_2\mathbf{1}\right)^T \mu - \mu_p = 0$$

$$-\frac{1}{2}\lambda_1\mu^T\Sigma^{-1}\mu + -\frac{1}{2}\lambda_2\mathbf{1}^T\Sigma^{-1}\mu = \mu_p$$

For the third equation:

$$w^T \mathbf{1} - 1 = 0$$

$$\left(-\frac{1}{2}\Sigma^{-1}\lambda_1\mu + -\frac{1}{2}\Sigma^{-1}\lambda_2\mathbf{1}\right)^T \mathbf{1} - 1 = 0$$

$$-\frac{1}{2}\lambda_1\mu^T\Sigma^{-1}\mathbf{1} + -\frac{1}{2}\lambda_2\mathbf{1}^T\Sigma^{-1}\mathbf{1} = 1$$

We find that we have a system of equations with two unknowns.

$$-\frac{1}{2}\lambda_1\mu^T\Sigma^{-1}\mu + -\frac{1}{2}\lambda_2\mathbf{1}^T\Sigma^{-1}\mu = \mu_p$$

$$-\frac{1}{2}\lambda_1\mu^T\Sigma^{-1}\mathbf{1} + -\frac{1}{2}\lambda_2\mathbf{1}^T\Sigma^{-1}\mathbf{1} = 1$$

To simplify, we group them in a matrix form. Define $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}_{2 \times 1}^T$; $U = \begin{bmatrix} \mu & \mathbf{1} \end{bmatrix}_{n \times 2}$ and $u = \begin{bmatrix} \mu_p & 1 \end{bmatrix}_{2 \times 1}^T$ and obtain:

$$-\frac{1}{2}U^T\Sigma^{-1}U\lambda = u$$

$$\lambda = -2[U^T\Sigma^{-1}U]^{-1}u$$

We substitute λ in w to find the efficient weights:

$$w = -\frac{1}{2}\Sigma^{-1}\lambda_1\mu + -\frac{1}{2}\Sigma^{-1}\lambda_2\mathbf{1} = -\frac{1}{2}\Sigma^{-1}U\lambda$$

$$w = -\frac{1}{2}\Sigma^{-1}U(-2[U^T\Sigma^{-1}U]^{-1}u) = \Sigma^{-1}U[U^T\Sigma^{-1}U]^{-1}u$$

We perform one last variable change, defining $M_{2 \times 2} = U^T \Sigma^{-1} U$.

$$w = \Sigma^{-1} U M^{-1} u \quad (2.8)$$

It is important to note that in the previous expression, the only undefined term is $u = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$, where u_p is the portfolio return. Therefore, the above equation indicates the weights for a set of assets given an expected return. The portfolios formed with these weights belong to Markowitz's frontier, which can be shown to have a hyperbolic shape.

2.- The Markowitz Frontier is a Hyperbola

To prove that the Markowitz frontier is a hyperbola, we can substitute the function that determines the frontier weights into the portfolio variance function:

$$\sigma^2 = w^T \Sigma w$$

$$\sigma^2 = (\Sigma^{-1} U M^{-1} u)^T \Sigma (\Sigma^{-1} U M^{-1} u)$$

$$\sigma^2 = u^T M^{-1} U^T \Sigma^{-1} \Sigma \Sigma^{-1} U M^{-1} u$$

Given that $A^{-1} A = I$ and knowing that $M = U^T \Sigma^{-1} U$, we simplify:

$$\sigma^2 = u^T M^{-1} U^T \Sigma^{-1} U M^{-1} u$$

$$\sigma^2 = u^T M^{-1} (U^T \Sigma^{-1} U) (U^T \Sigma^{-1} U)^{-1} u$$

$$\sigma^2 = u^T M^{-1} u$$

Expanding and simplifying for M :

$$M = U^T \Sigma^{-1} U = \begin{bmatrix} \mu^T \\ \mathbf{1}^T \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \mu \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mu^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mathbf{1} \\ \mathbf{1}^T \Sigma^{-1} \mu & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix}$$

To calculate the inverse of M given that it is a 2×2 matrix, we can use the rule that states for a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ with $\det(A) = ad - bc$. Note that M is a symmetric matrix, so $b = c$. We solve M^{-1} :

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} \mathbf{1}^T \Sigma^{-1} \mathbf{1} & -\mathbf{1}^T \Sigma^{-1} \mu \\ -\mathbf{1}^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mu \end{bmatrix}$$

With this, we can proceed to solve for σ^2 :

$$\sigma^2 = \begin{bmatrix} \mu_p & 1 \end{bmatrix} \left(\frac{1}{\det(M)} \begin{bmatrix} \mathbb{1}^T \Sigma^{-1} \mathbb{1} & -\mathbb{1}^T \Sigma^{-1} \mu \\ -\mathbb{1}^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mu \end{bmatrix} \right) \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

$$\sigma^2 = \frac{1}{\det(M)} [(\mathbb{1}^T \Sigma^{-1} \mathbb{1}) \mu_p^2 - 2(\mathbb{1}^T \Sigma^{-1} \mu) \mu_p + (\mu^T \Sigma^{-1} \mu)]$$

To simplify, we substitute variables by defining $s_{11} = \mathbb{1}^T \Sigma^{-1} \mathbb{1}$, $s_{1\mu} = \mathbb{1}^T \Sigma^{-1} \mu$, $s_{\mu\mu} = \mu^T \Sigma^{-1} \mu$ and $d = \det(M)$:

$$\sigma^2 = \frac{s_{11} \mu_p^2 - 2s_{1\mu} \mu_p + s_{\mu\mu}}{d}$$

$$\sigma = f(\mu_p) = \sqrt{\frac{s_{11} \mu_p^2 - 2s_{1\mu} \mu_p + s_{\mu\mu}}{d}} \quad (2.9)$$

As we can see, the risk can be expressed as a function of the portfolio return, and this relationship takes the form of a hyperbola, as shown in the graph 2.3. The portfolios that lie on the hyperbola and above the minimum variance portfolio represent the mean-variance efficient portfolios. These are the portfolios for which there is no better return for a given level of risk.

3.- Capital Market Line

Finally, to introduce the concept of the Sharpe Ratio, we will derive the capital market line analytically. This line is the tangent line to the efficient frontier and represents the combination of a risk-free asset with an efficient investment portfolio. This concept can be derived from the same optimization problem of the efficient frontier presented previously but adding a risk-free asset, which we will define as r_f :

$$\min \sigma^2 : w^T \Sigma w \quad (2.10)$$

$$\text{subject to } w^T \mathbb{1} + w_{r_f} = 1$$

$$w^T \mu + w_{r_f} r_f = \mu_p$$

Where w_{r_f} is the scalar representing the weight of the risk-free rate within the portfolio. We can simplify the constraints by introducing the first constraint into the second one, given that $w_{r_f} = 1 - w^T \mathbb{1}$:

$$\mu_p = w^T \mu + w_{r_f} r_f$$

$$\mu_p = w^T \mu + (1 - w^T \mathbb{1}) r_f$$

$$\mu_p = w^T \mu + r_f - w^T r_f \mathbf{1}$$

$$\mu_p - r_f = w^T \mu - w^T r_f \mathbf{1}$$

$$\mu_p - r_f = w^T (\mu - r_f \mathbf{1})$$

With this modification, the optimization problem is as follows:

$$\begin{aligned} \min \sigma^2 : \quad & w^T \Sigma w \\ \text{subject to} \quad & \mu_p - r_f = w^T (\mu - r_f \mathbf{1}) \end{aligned}$$

We derive the Lagrangian:

$$L = w^T \Sigma w + \lambda (w^T (\mu - r_f \mathbf{1}) + r_f - \mu_p)$$

$$\nabla L = \begin{bmatrix} 2\Sigma w + \lambda(\mu - r_f \mathbf{1}) \\ w^T (\mu - r_f \mathbf{1}) + r_f - \mu_p \end{bmatrix}$$

Setting the first element of the matrix to 0 and solving for w :

$$2\Sigma w + \lambda(\mu - r_f \mathbf{1}) = 0$$

$$w = -\frac{1}{2} \Sigma^{-1} \lambda (\mu - r_f \mathbf{1})$$

To find the value of λ , we substitute w in the second equation of the gradient:

$$w^T (\mu - r_f \mathbf{1}) + r_f - \mu_p$$

$$\left(-\frac{1}{2} \Sigma^{-1} \lambda (\mu - r_f \mathbf{1})\right)^T (\mu - r_f \mathbf{1}) + r_f - \mu_p = 0$$

$$-\frac{1}{2} \lambda (\mu - r_f \mathbf{1})^T \Sigma^{-1} (\mu - r_f \mathbf{1}) = \mu_p - r_f$$

$$\lambda = -\frac{2(\mu_p - r_f)}{(\mu - r_f \mathbf{1})^T \Sigma^{-1} (\mu - r_f \mathbf{1})}$$

Substituting λ in w :

$$w = -\frac{1}{2} \left(\frac{-2(\mu_p - r_f)}{(\mu - r_f \mathbf{1})^T \Sigma^{-1} (\mu - r_f \mathbf{1})} \right) \Sigma^{-1} (\mu - r_f \mathbf{1})$$

$$w = \frac{(\mu_p - r_f)\Sigma^{-1}(\mu - r_f\mathbb{1})}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})}$$

Substituting w into the equation for σ^2 :

$$\begin{aligned}\sigma^2 &= w^T\Sigma w \\ \sigma^2 &= \left(\frac{(\mu_p - r_f)\Sigma^{-1}(\mu - r_f\mathbb{1})}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \right)^T \Sigma \frac{(\mu_p - r_f)\Sigma^{-1}(\mu - r_f\mathbb{1})}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \\ \sqrt{\sigma^2} &= \sqrt{\frac{(\mu_p - r_f)^2}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})}} \\ \sigma &= \frac{(\mu_p - r_f)}{\sqrt{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})}}\end{aligned}$$

In this case, we seek to express the return as a function of volatility, so we solve for μ_p :

$$\mu_p = r_f \pm \sigma \sqrt{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})}$$

Since we are only interested in those portfolios that are tangent to the efficient frontier, we consider only the positive solution in the formula above.

$$\mu_p = r_f + \sigma \sqrt{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \quad (2.11)$$

With this equation, we can calculate the tangent line to the efficient frontier, known as the Capital Market Line (CML). This line considers investment in a risk-free asset and an efficient portfolio. Here, we observe two key concepts: first, the portfolio return μ_p increases as volatility increases, implying the risk-return relationship: higher risk, higher return.

On the other hand, solving for the slope of the equation, we find that $\sqrt{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} = \frac{\mu_p - r_f}{\sigma}$, where the equation on the right is the Sharpe Ratio, a concept we will analyze in the next section. We can conclude that the slope of the tangent line to the efficient frontier is the Sharpe Ratio; thus, the maximum Sharpe Ratio is on the efficient frontier, and by calculating it, we can determine the CML.

2.1.4 Maximum Sharpe Efficient Portfolio

The maximum Sharpe portfolio¹³ is the one that maximizes the ratio between return and risk. This ratio is measured through the Sharpe ratio, which is a metric used to evaluate the relationship between the excess return over the risk-free rate of an investment and the additional risk assumed.

$$SR = \frac{\mu_p - r_f}{\sigma_p} \quad (2.12)$$

Where:

- μ_p : Portfolio return
- r_f : Risk-free rate
- σ_p : Portfolio volatility

The numerator of the formula is known as the excess return over the risk-free rate. This risk-free rate is a key reference in the financial domain and is generally known a priori by the investment manager, as the central bank of the reference country regularly publishes it. The risk-free rate represents the minimum return an investor can expect to earn without taking on additional risk since, in theory, investing in government-issued securities is considered free of default risk.

Then again, portfolio return and volatility are the same concepts defined previously, where the former corresponds to the expected portfolio return, and the latter represents the portfolio's standard deviation. Using the previously defined formula for μ_p and σ_p , we can rewrite it in matrix form:

$$SR = \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}}$$

This ratio indicates how many units of return we receive above the risk-free rate for each unit of risk taken. The larger it is, the better the portfolio's risk-return relationship. The optimization problem is expressed as:

$$\begin{aligned} \max SR : & \frac{\mu_p - r_f}{\sigma_p} \\ \text{subject to :} & \mathbf{1}^T w = 1 \end{aligned}$$

To solve the problem analytically¹⁴, we first simplify it using the

¹³ William F. Sharpe. *Portfolio Theory and Capital Markets*. McGraw-Hill, 1966

¹⁴ Robert Merton. An analytic derivation of the efficient portfolio frontier. *Cambridge University Press*, Vol. 7, pp. 1851-1872, 1972

previous solution for σ_p . We found that portfolio volatility can be written as a function of return $\sigma_p = f(\mu_p) = \sqrt{\frac{s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu}}{d}}$. The determinant d does not depend on μ_p , so we can rewrite $\sigma_p = f(\mu_p) = \sqrt{s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu}}$. Since we initially seek to solve for μ_p , we can omit the constraint, leaving our optimization problem as follows:

$$\max \mu_p : \frac{\mu_p - r_f}{\sqrt{s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu}}}$$

We take the derivative with respect to μ_p and set it to zero:

$$\frac{\partial}{\partial \mu_p} = \frac{1}{\sqrt{s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu}}} - \frac{1}{2} \frac{(2s_{11}\mu_p - 2s_{1\mu})(\mu_p - r_f)}{(s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu})^{\frac{3}{2}}} = 0$$

To eliminate the roots, we multiply the entire expression by $(s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu})^{\frac{3}{2}}$ and simplify:

$$s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu} - \frac{1}{2}(2s_{11}\mu_p - 2s_{1\mu})(\mu_p - r_f) = 0$$

$$s_{11}\mu_p^2 - 2s_{1\mu}\mu_p + s_{\mu\mu} - s_{11}\mu_p^2 + s_{11}\mu_p r_f + s_{1\mu}\mu_p - s_{1\mu}r_f = 0$$

$$-s_{1\mu}\mu_p + s_{11}\mu_p r_f + s_{\mu\mu} - s_{1\mu}r_f = 0$$

Solving for μ_p :

$$\mu_p^{max} = \frac{s_{1\mu}r_f - s_{\mu\mu}}{s_{11}r_f - s_{1\mu}}$$

To find the optimal weights at the maximum Sharpe ratio point, we substitute μ_p^{max} into the previously derived formula for the efficient weights on the Markowitz frontier, i.e., in $w = \Sigma^{-1}UM^{-1}u$, obtaining:

$$w = \Sigma^{-1} \begin{bmatrix} \mu & \mathbf{1} \end{bmatrix} \left[\frac{1}{\det(M)} \begin{bmatrix} s_{11} & -s_{1\mu} \\ -s_{1\mu} & s_{\mu\mu} \end{bmatrix} \right] \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

We substitute $\det(M) = s_{11}s_{\mu\mu} - s_{1\mu}^2$ y $\mu_p^{max} = \frac{s_{1\mu}r_f - s_{\mu\mu}}{s_{11}r_f - s_{1\mu}}$:

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \begin{bmatrix} \mu & \mathbf{1} \end{bmatrix} \begin{bmatrix} s_{11} & -s_{1\mu} \\ -s_{1\mu} & s_{\mu\mu} \end{bmatrix} \begin{bmatrix} \frac{s_{1\mu}r_f - s_{\mu\mu}}{s_{11}r_f - s_{1\mu}} \\ 1 \end{bmatrix}$$

Simplifying:

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \begin{bmatrix} \mu & \mathbb{1} \end{bmatrix} \begin{bmatrix} \frac{s_{11}s_{1\mu}r_f - s_{11}s_{\mu\mu}}{s_{11}r_f - s_{1\mu}} - s_{1\mu} \\ \frac{s_{\mu\mu}s_{1\mu} - s_{1\mu}^2r_f}{s_{11}r_f - s_{1\mu}} + s_{\mu\mu} \end{bmatrix}$$

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \begin{bmatrix} \mu & \mathbb{1} \end{bmatrix} \begin{bmatrix} \frac{s_{11}s_{1\mu}r_f - s_{11}s_{\mu\mu} - s_{1\mu}s_{11}r_f + s_{1\mu}^2}{s_{11}r_f - s_{1\mu}} \\ \frac{s_{\mu\mu}s_{1\mu} - s_{1\mu}^2r_f + s_{\mu\mu}s_{11}r_f - s_{\mu\mu}s_{1\mu}}{s_{11}r_f - s_{1\mu}} \end{bmatrix}$$

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \begin{bmatrix} \mu & \mathbb{1} \end{bmatrix} \begin{bmatrix} \frac{-s_{11}s_{\mu\mu} + s_{1\mu}^2}{s_{11}r_f - s_{1\mu}} \\ \frac{rf(s_{\mu\mu}s_{11} - s_{1\mu}^2)}{s_{11}r_f - s_{1\mu}} \end{bmatrix}$$

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \left[\frac{-\mu(s_{11}s_{\mu\mu} + s_{1\mu}^2)}{s_{11}r_f - s_{1\mu}} + \frac{rf\mathbb{1}(s_{\mu\mu}s_{11} - s_{1\mu}^2)}{s_{11}r_f - s_{1\mu}} \right] \frac{-1}{-1}$$

$$w = \frac{1}{s_{11}s_{\mu\mu} - s_{1\mu}^2} \Sigma^{-1} \left[\frac{\mu(s_{11}s_{\mu\mu} + s_{1\mu}^2)}{s_{1\mu} - s_{11}r_f} - \frac{rf\mathbb{1}(s_{\mu\mu}s_{11} - s_{1\mu}^2)}{s_{1\mu} - s_{11}r_f} \right]$$

$$w = \Sigma^{-1} \left[\frac{\mu}{s_{1\mu} - s_{11}r_f} - \frac{rf\mathbb{1}}{s_{1\mu} - s_{11}r_f} \right] = \Sigma^{-1} \left[\frac{\mu - rf\mathbb{1}}{s_{1\mu} - s_{11}r_f} \right]$$

Since $s_{1\mu} = \mathbb{1}^T \Sigma^{-1} \mu$ and $s_{11} = \mathbb{1}^T \Sigma^{-1} \mathbb{1}$, we obtain:

$$w = \frac{\Sigma^{-1}(\mu - rf\mathbb{1})}{\mathbb{1}^T \Sigma^{-1}(\mu - rf\mathbb{1})} \quad (2.13)$$

This formula represents the optimal weights for the maximum Sharpe ratio portfolio. With these weights, we can calculate the maximum Sharpe ratio using $SR = \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} = \frac{\mu_p^{\max} - r_f}{\sigma_p}$, where μ_p^{\max} represents the return and σ_p represents the volatility.

As demonstrated earlier, this ratio is the slope of the Capital Market Line (CML). Since the CML is the tangent line to the efficient frontier, the maximum Sharpe ratio portfolio is on this frontier. This portfolio is often recommended for investors with a higher risk tolerance, as maximizing the risk-return ratio results in a strategy with greater volatility compared to the minimum variance strategy.

2.1.5 Backtesting

Backtesting is a portfolio simulation method that enables investors to evaluate the effectiveness of an investment strategy statistically. This process involves applying the approach to historical market data and constructing the portfolio's returns over a specified time horizon to analyze its past performance¹⁵.

Given the nature of stock market assets, their historical information is publicly available, making it relatively easy to obtain (depending on the base market) through information providers like *Yahoo Finance*. The most critical parameters in backtesting are the time horizon and the frequency with which the portfolio is to be rebalanced¹⁶—this is in cases where the strategy is tested with changes in weights through re-optimization-

Backtesting allows to:

- Validate the strategy's effectiveness.
- Fine-tune the strategy's parameters.
- Predict the future performance, assuming that portfolio returns follow a constant mean and variance over the long term.

Conducting a simulation of an investment strategy is always recommended, as it helps to understand the nuances of the strategy and calculate expected performance metrics¹⁷. These performance metrics include annual return, volatility, and the Sharpe ratio, among others.

An essential part of the backtesting process is comparing the results against a benchmark¹⁸ to determine whether the investment strategy is achieving superior, equal, or inferior returns and volatilities compared to the market. Since clients of an asset manager typically pay a commission for the expected outperformance of the strategy over the market, it is recommended that non-professional investors invest in the benchmark if this outperformance does not materialize.

A common mistake in this methodology is using the same data for both portfolio optimization and simulation¹⁹. While this practice is common and acceptable in the financial context, and a section of this thesis will address it accordingly, the recommended approach is to optimize using a time window t and then simulate the performance in the subsequent window $t + 1$. This separation not only enhances the robustness of the results but also provides a more realistic evaluation of the model's ability to adapt to new market conditions.

¹⁵ Leo Wong. Introductory backtesting notes for quantitative trading strategies. *Hong Kong University of Science and Technology*, 2019

¹⁶ Portfolio rebalancing is a process in which an investor adjusts the weights of their portfolio to maintain its risk level according to their tolerance.

Yu Zhang, Harshdeep Ahluwalia, Allison Ying, Michael Rabinovich, and Aidan Geysen. Portfolio rebalancing. *Vanguard*, 2022

¹⁷ Ernest Chan. *Quantitative Trading*. Wiley, 2021

¹⁸ A benchmark is a market index that shows how the stock market has generally performed over a given period.

Benjamin Graham. *The Intelligent Investor*. Harper Collins, 1949

¹⁹ Guillaume Coqueret and Tony Guida. *Machine Learning for Factor Investing*. Chapman and Hall, 2020

2.1.6 Performance Metrics

Although variance and volatility are classic metrics for estimating the risk of a portfolio -i.e. deviations from the mean of returns- it is possible to further break down this concept into two distinct components: positive deviations (upside risk) and negative deviations (downside risk). These correspond to the risks associated with potential gains and losses, respectively²⁰. The formal definition of these types of risk are as follows²¹:

- **Upside Risk:** The standard deviation (volatility) of returns that lead to capital gains. In a portfolio with long positions, this refers to returns above zero.
- **Downside Risk:** The standard deviation (volatility) of returns that result in capital losses. In a portfolio with long positions, this refers to returns below zero.

These risks can be calculated by simply separating returns below and above zero and then calculating the standard deviation for each subset, which is often know as the non-parametric or historical method²². More formally, given that financial returns tend to follow a normal probability distribution, downside risk and upside risk can also be calculated parametrically with the following formulas, which mathematically reflect the same logic as non-parametric methods²³.

$$\sigma_u = \int_0^{\infty} F(x)dx \quad (2.14)$$

$$\sigma_d = \int_{-\infty}^0 F(x)dx \quad (2.15)$$

These integrals represent upside and downside risk, respectively. Empirically, these can be calculated by separating the returns as mentioned earlier. For an investment portfolio, it is generally preferable to have a higher positive semi-deviation (upside risk) compared to negative semi-deviation (downside risk). The greater the ratio between upside and downside risk, the better the portfolio's profile. This ratio is known as the Omega Ratio²⁴, which can be calculated as follows:

$$\Omega = \frac{\sigma_u}{\sigma_d} \quad (2.16)$$

The Omega Ratio indicates how many units of upside risk we are receiving per unit of downside risk taken. A ratio greater than one

²⁰ Ron Dembo and Daniel Stoffman. *Upside and Downside: Simple Rules of Risk Management*. Doubleday Canada, 2006

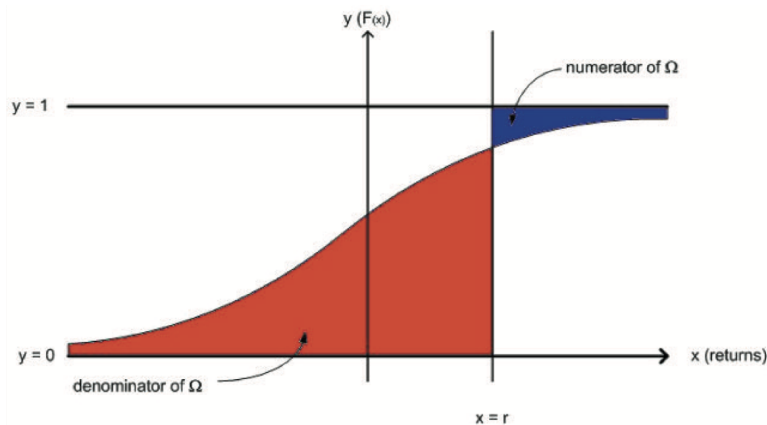
²¹ Alexander McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton University, 2005

²² Karen Horcher. *Essentials of Financial Risk Management*. John Wiley and Sons, 2005

²³ Ranjan Bhaduri and Bryon Kaneshige. *Risk management: Taming the tail. Benefits and Pensions Monitor*, 2005

²⁴ Ranjan Bhaduri and Bryon Kaneshige. *Risk management: Taming the tail. Benefits and Pensions Monitor*, 2005

implies that the total risk of a portfolio is skewed more towards positive returns than negative ones. Hence, a rational investor would prefer a strategy with a higher Omega Ratio. Below is a graphical representation of the Omega Ratio.



It is important to note that the graph assumes a threshold " r " where downside risk is measured. In this thesis, we adopt a threshold of zero, focusing specifically on the standard deviation of returns that fall below this point. Essentially, we will be examining negative returns. Alternatively, we could define the threshold based on the returns of a benchmark, allowing us to assess the risk associated with failing to achieve returns that surpass this benchmark²⁵.

Another crucial metric for evaluating portfolio performance is the Sortino Ratio, which is a refinement of the Sharpe Ratio. The Sortino Ratio improves upon the Sharpe Ratio as a portfolio's performance metric by replacing total portfolio risk (volatility) with downside risk, as only downside risk represents potential losses for an investment portfolio²⁶. The Sortino Ratio is given by the following formula:

$$S = \frac{\mu_p - r_f}{\sigma_d} \quad (2.17)$$

This ratio reflects how many units of return above the risk-free rate are earned per unit of downside risk. It follows the logic that portfolio managers are primarily concerned with penalizing negative returns, as it represents the actual expected losses²⁷. For this reason, the Sortino Ratio is considered a more accurate measure of a portfolio manager's performance. These metrics will be employed in this thesis to assess the behavior of the strategies during backtesting.

Figure 2.2: Graphical Representation of the Omega Ratio

²⁵ Pamela Peterson. *CFA Level I Quantitative Methods: Statistical Measures of Asset Returns*. CFA Institute, 2024

²⁶ Deborah Kidd. The sortino ratio. *CFA Institute*, 2012

²⁷ Frank Sortino. Performance measurement in a downside risk framework. *Journal of Investing*, Vol. 3, pp. 59-65, 1994

2.2 Optimization

Optimization is the process of finding the *best* solution to a given problem within a set of constraints. These constraints can be of various natures, such as physical, budgetary, or regulatory limitations, and must be satisfied for the solution to be feasible²⁸.

Given a convex optimization problem, the set of all possible solutions that satisfy the constraints is called the feasible region. Formally, this region consists of all points in the variable space that meet the equality and inequality constraints. The convexity of a problem ensures that there is a unique optimal solution within the feasible region²⁹. This optimal solution can be either a minimum or maximum point, depending on the problem statement.

The optimization methodology includes analytical and numerical techniques, which are used to solve both linear and nonlinear problems, aiming to achieve the most efficient solution within the feasible solution space. A trivial optimization problem is defined as follows³⁰:

$$\min f(x)$$

Subject to:

$$h(x) = 0$$

$$g(x) \leq 0$$

Additionally:

$$x \in X$$

Where x represents the decision variables³¹, $f(x)$ is a function that maps x from \mathbb{R}^n to \mathbb{R} , $h(x) = 0$ is a set of equality constraints, $g(x) \leq 0$ is a set of inequality constraints, and $X \in \mathbb{R}^n$ is a convex set.

It should be noted that optimization problems for efficient minimum variance portfolios (2.1.2) and maximum Sharpe ratio portfolios (2.1.4) follow this form, where our decision variables are the portfolio weights defined as w . In a more general sense, optimization involves systematically choosing the best option from a set of available alternatives. This process can be applied to various domains, including engineering, economics, logistics, and computer science. The goal is to maximize or minimize a specific objective function, which represents any quantitative measure, for example, performance, cost, or efficiency.

²⁸ Michael Bierlaire. *Optimization: Principles and Algorithms*. EPFL Press, 2015

²⁹ Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004

³⁰ Michael Bierlaire. *Optimization: Principles and Algorithms*. EPFL Press, 2015

³¹ Decision variables represent the components of a system that can be and are desired to be modified.

Singiresu Rao. *Engineering Optimization*. John Wiley and Sons, 2009

2.2.1 Regularization

Regularization is a mathematical method used to penalize the outcomes of an optimization problem with a specific objective³². This technique involves explicitly adding a term—another function—to the objective function, which allows for controlling the outcomes of the optimization problem to some extent.

This technique is instrumental in statistical and machine learning models, where the intent is to improve the model's generalization. In short, the goal of regularization is to penalize an objective function based on a second given function, helping control the model's complexity and prevent overfitting³³.

To illustrate this, consider a generic quadratic optimization problem:

$$\text{minimize } \|Ax - b\|^2$$

In this case, A is a matrix of coefficients, x is the vector with decision variables, and b is the vector with the ordinates. The objective function $\|Ax - b\|$ represents the L2 norm. However, by incorporating regularization, we add a term that minimizes the quadratic sum of the decision variables:

$$\text{minimize } \|Ax - b\|^2 + \lambda\|x\|^2$$

Here, λ is a parameter specific to any regularization problem that controls the degree of penalty imposed by the regularization function. This additional term, at least in this context, penalizes solutions where the quadratic sum of the decision variable x is large. By adjusting λ , we can balance between minimizing the objective function and controlling the magnitude of the model's coefficients, resulting in a solution that is optimal on both fronts³⁴.

This method of regularization is applied in various settings to solve multi-objective problems. Considering more than one objective allows an analyst to manipulate the results to arrive at a solution based on different functions³⁵. For example, in the context of machine learning, regularization can help standardize a regression's weights within a specific range, while in financial optimization, it can balance return and risk.

³² Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004

³³ Vincent Vandembussche. *The Regularization Cookbook: Explore practical recipes to improve the functionality of your ML models*. Packt, 2023

³⁴ Christopher M. Bishop. *Pattern Recognition and Machine Learning*. Springer, 2006

³⁵ Vincent Vandembussche. *The Regularization Cookbook: Explore practical recipes to improve the functionality of your ML models*. Packt, 2023

2.2.2 Multiobjective Optimization

In multi-objective optimization problems, we are tasked with simultaneously optimizing several objective functions that conflict with one another.

This means that achieving the optimal point in one objective function involves compromising performance in the other functions. In these problems, a set of equally valid solutions is obtained, but with different performances in each objective of the optimization problem³⁶.

The canonical multi-objective optimization problem can be written as follows:

$$\begin{aligned} &\text{minimize} && f(x; p) = (f_1(x; p), \dots, f_r(x; p))^T \\ &\text{subject to:} && g_j(x; p) \leq 0, \quad j = 1, \dots, m, \\ &&& h_l(x; p) = 0, \quad l = 1, \dots, p, \\ &&& x_i \in [x_i^L, x_i^U], \quad i = 1, \dots, n, \end{aligned}$$

In this case, there are r objective functions denoted as $f_i(x; p)$, where x represents the decision variables and p the parameter of the problem. Additionally, there are m inequality constraints denoted as $g_j(x; p)$ and p equality constraints, denoted as $h_l(x; p)$.

The goal is to find the feasible region of solutions for and that satisfy all conditions, allowing the identification of optimal values that are *valid* for all objective functions. To explain this better, we introduce the concept of a Pareto optimal point. Consider two points x and y within the feasible region. A point x, y is considered a Pareto optimal point if and only if it meets the following conditions³⁷:

1. Feasibility: Both points x and y must be feasible, meaning they must satisfy all the constraints of the problem.
2. Dominance: There is no point y in the feasible region such that $F(y)$ improves all objectives compared to $F(x)$ or is at least equal in all objectives and better in at least one.

In simple terms, for a point x^{po} (Pareto optimal), there is no other point y such that $F(y) \leq F(x^{po})$ in all objectives, with $F(y) \neq F(x^{po})$ in at least one of them. This means that for the given value of x^{po} , no value of y can be found that is strictly better in all considered objectives. Therefore, this point is regarded as a Pareto optimal point and is part of the feasible region.

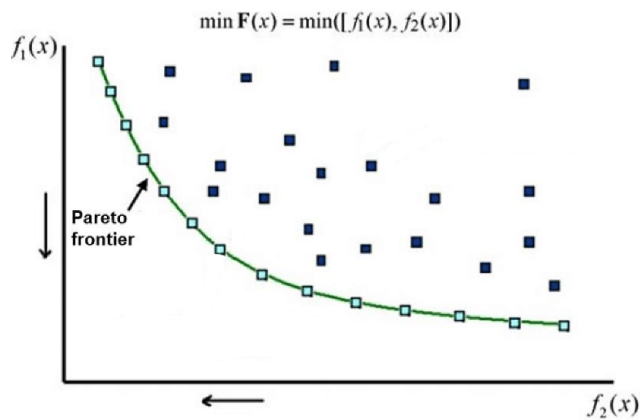
³⁶ André A. Keller. *Multi-Objective Optimization in Theory and Practice I: Classical Methods*. Bentham eBooks, 2017

³⁷ Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004

2.2.3 Pareto Front

The set of Pareto optimal points for a multi-objective problem is called the Pareto Front³⁸. Each solution in the Pareto Front represents a trade-off between objectives, meaning that improving performance in one necessarily involves a reduction in the performance of another.

Although all solutions in the Pareto Front are considered optimal, selecting the ideal solution should be based on evaluating these trade-offs and how they align with the analyst's preferences and priorities. Below is a graphical example of a Pareto Front for a minimization problem with two objective functions³⁹.



The figure illustrates the Pareto Front as a region containing the set of optimal solutions. Moving along this front can improve one objective function at the cost of worsening the other, demonstrating the trade-offs between the performances of the objectives. For example, selecting a Pareto point on the lower right side yields poor performance on the second objective function but excellent performance on the first. It is conversely, choosing a point on the upper left side results in exceptional performance on the first objective function but at the cost of poor performance on the second.

Connecting this concept with the portfolio asset allocation problem, the figure in section 2.1.3 shows the efficient frontier in mean-variance, which is a Pareto Front for a set of assets where the objective functions are risk and return. This figure shows all possible combinations for the given assets. The upper edge of the hyperbola is known as the mean-variance efficient frontier. For these portfolios, it is not possible to achieve a higher return for the level of risk taken, making these portfolios *optimal* on the mean-variance front.

³⁸ Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004

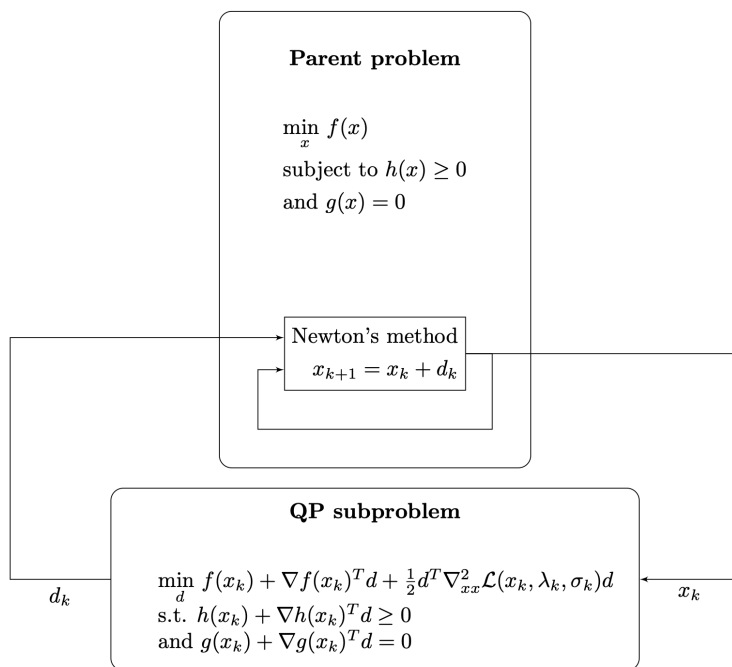
³⁹ Marialaura Di Somma. *Optimal Operation Planning of Distributed Energy Systems Through Multi-Objective Approach*. PhD thesis, University of Naples, 2016

Figure 2.3: Pareto Front for a trivial minimization problem

Within this efficient frontier, there are multiple optimal portfolios, including the minimum variance portfolio and the maximum Sharpe portfolio. While both are optimal, the choice between them depends on the investor's preferences: the first one minimizes risk, while the second maximizes the relation between return and risk. This demonstrates the importance of Pareto Fronts in portfolio optimization, where each optimal asset combination offers a different balance between the objective functions, allowing investors to select the option that best matches their objectives and risk tolerance.

2.2.4 SLSQP

The SLSQP (Sequential Least Squares Quadratic Programming) method is a numerical optimization algorithm proposed by Dieter Kraft⁴⁰. It seeks to solve nonlinear problems with constraints by modifying Newton's algorithm and incorporating a quadratic optimization subproblem⁴¹. Below is a schematic of the method's routine:



This algorithm is considered a "Quasi-Newton" method as it modifies the classical Newton algorithm⁴² by adding the factor d_k , referred to by Kraft as the *direction*. As shown in the figure above, the direction is calculated by solving a quadratic optimization sub-problem, which takes into account the inequality and equality constraints of the main problem to determine the direction, transferring this information to the iterative method.

⁴⁰ Dieter Kraft. A software package for quadratic programming. *Institut für Dynamik der Flugsysteme*, Vol. 1, pp. 3-33, 1988

⁴¹ For further investigation on these concepts, the following reference is provided to the reader.

Michael Bierlaire. *Optimization: Principles and Algorithms*. EPFL Press, 2015

Figure 2.4: Schematic of the SLSQP Algorithm proposed by Kraft

⁴² Reference for the reader.

Isaac Newton. *Philosophiæ naturalis principia mathematica*. 1687; and Joseph Raphson. *Analysis aequationum universalis*. *Royal Society of London*, Vol. 44, pp. 151-167, 1690

Sequential quadratic programming algorithms are widely recognized for their efficiency in solving nonlinear optimization problems with constraints. These methods are precious due to their ability to handle complex problems with a significant number of variables and constraints, optimizing both computation time and convergence speed. Thanks to their iterative approach, these algorithms can effectively approximate the optimal solution by solving a series of sequential quadratic problems, adapting to the problem structure, and improving efficiency with each iteration⁴³.

Furthermore, sequential quadratic programming algorithms offer significant advantages in terms of stability and robustness compared to other nonlinear optimization methods. Their ability to quickly converge to near-optimal solutions is advantageous in practical applications where execution time is critical. This efficiency has been extensively documented in the literature, demonstrating that these algorithms are a fundamental tool in the field of optimization⁴⁴.

2.2.5 Hyperparameter Optimization

A hyperparameter is a predefined parameter used to guide and control the behavior of an algorithm set before the learning process begins. Unlike model parameters, which are adjusted during the training process, hyperparameters are configured beforehand and can significantly impact the performance of the model⁴⁵. Proper tuning of hyperparameters leads to better model accuracy and improved generalization to new data.

Various algorithms have been developed over the years to find the best hyperparameters in an optimization problem. In this work, we employ the *Tree-structured Parzen Estimator Approach (TPE)*, which models a problem using Bayesian optimization theory⁴⁶. Unlike a Gaussian process that models $p(y|x)$, TPE models both $p(y|x)$ and $p(y)$, where y represents the hyperparameter and x represents the decision variables of the problem.

The TPE algorithm works by constructing two models: one for the objective function given the hyperparameters and one for the distribution of the hyperparameters themselves. It separates the search space into good and bad regions based on the performance of sampled hyperparameters. Then, it guides the search towards the promising regions by sampling more frequently from them⁴⁷. This approach tends to converge to the optimum in less time compared to other methods. The Python library Optuna implements this algorithm for hyperparameter optimization⁴⁸.

⁴³ Willi Hock and Klaus Schittkowsky. Test examples for nonlinear programming codes. *University of Bayreuth*, 1981

⁴⁴ Willi Hock and Klaus Schittkowsky. Test examples for nonlinear programming codes. *University of Bayreuth*, 1981

⁴⁵ Li Yang and Abdallah Shami. On hyperparameter optimization of machine learning algorithms. *Neurocomputing*, Vol. 415, pp. 295–316, 2020

⁴⁶ Reference for the reader. Roman Garnett. *Bayesian Optimization*. Cambridge University Press, 2023

⁴⁷ James Bergstra, Rémi Bardenet, Yoshua Bengio, and Balázs Kégl. Algorithms for hyper-parameter optimization. *Harvard University*, 2011

⁴⁸ Takuya Akiba, Shotaro Sano, Toshihiko Yanase, Takeru Ohta, and Masanori Koyama. Optuna: A next-generation hyperparameter optimization framework. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2019

2.3 Market Liquidity Risk

Market liquidity risk is the non-diversifiable risk that a financial position cannot be easily liquidated in the short term without significantly affecting its value due to a lack of market depth⁴⁹. This arises when there are not enough buyers or sellers in the market to accommodate large transactions without causing substantial price fluctuations.

Consequently, liquidity risk is directly related to market efficiency. In efficient markets, participants can execute transactions without incurring significant liquidity costs. However, in inefficient markets, liquidity costs can considerably impact the performance of an investment portfolio. Therefore, in such markets, this cost must be a central factor in constructing the optimal portfolio⁵⁰.

While this type of risk is complex to quantify since it depends on the total value of the portfolio and the liquidity of its constituent assets, there are various methodologies to estimate it. The market convention is to take the proportion of the average daily volume over the last three months (*ADV*)⁵¹. The formula is as follows:

$$N_{days} = \frac{N_{shares}}{ADV}$$

Where:

- N_{shares} : Number of shares of a given asset
- *ADV*: Average daily volume of the asset over the last three months

This formula serves as an indicator of liquidity risk, representing the number of days required to liquidate a position in a financial asset fully. The higher the proportion of the position in the *ADV*, the higher the liquidity cost⁴². This ratio has been used in numerous practical studies, for example, for estimating liquidation times⁵².

In order to solve the portfolio problem considering liquidity risk, we need to have the asset weights in the given function rather than the number of shares so that we can optimize them. The following formula facilitates this process⁵³:

$$N_{shares} = \frac{C}{P} w = C w \frac{1}{P}$$

Where:

- *C*: Capital or total value of the portfolio

⁴⁹ Philippe Jorion. *Financial Risk Manager Handbook*. Wiley Finance, 2011

⁵⁰ Viral Acharya and Lasse Heje. Economics with market liquidity risk. *CBS Research*, Vol. 8, pp. 111-125, 2019

⁵¹ Thierry Roncalli and Guillaume Weisang. Asset management and systemic risk. *Secretariat of Financial Stability Board*, 2015

⁵² Autorité des marchés financiers. The use of stress tests as part of risk management. *AMF France*, 2017

⁵³ Ludwig Chincarini and Daehwan Kim. *Quantitative Equity Portfolio Management*. McGraw Hill, 2006

- w : Weight of the asset in the portfolio
- P : Current price of the asset

Using this, we can rewrite the equation to account for the asset's weight within an investment portfolio:

$$N_{days} = \frac{C w \frac{1}{P}}{ADV} \quad (2.18)$$

This equation evaluates liquidity risk on an individual level, i.e., for a single asset, as liquidity risk is not diversifiable. This means that, unlike other types of risk that can be mitigated through diversification, liquidity risk must be calculated and managed separately for each asset in the portfolio⁵⁴. This calculation is crucial for financial decision-making, as it helps to ensure that liquidity obligations can be met when necessary.

⁵⁴ Kleopatra Nikolaou. Liquidity concepts. *European Central Bank*, 2009

2.4 Mexican Stock Exchange

A stock exchange is a financial entity where market operations are conducted, aimed at facilitating transactions, fostering market development, and promoting its expansion and competitiveness⁵⁵. In Mexico, this is the *Bolsa Mexicana de Valores (BMV)*, whereas in the United States, the central stock exchange is the *New York Stock Exchange (NYSE)*.

⁵⁵ Maria Domínguez. Análisis de la bolsa mexicana de valores, 2020

On the other hand, a liquid market is defined as *a market where transactions can be executed without incurring any transaction costs*, and one of its main characteristics is that *the transaction volumes are generally high*⁵⁶.

⁵⁶ Michael Fleming. Measuring treasury market liquidity. *Federal Reserve Economic Policy Review*, Vol. 9, pp. 83-108, 2003

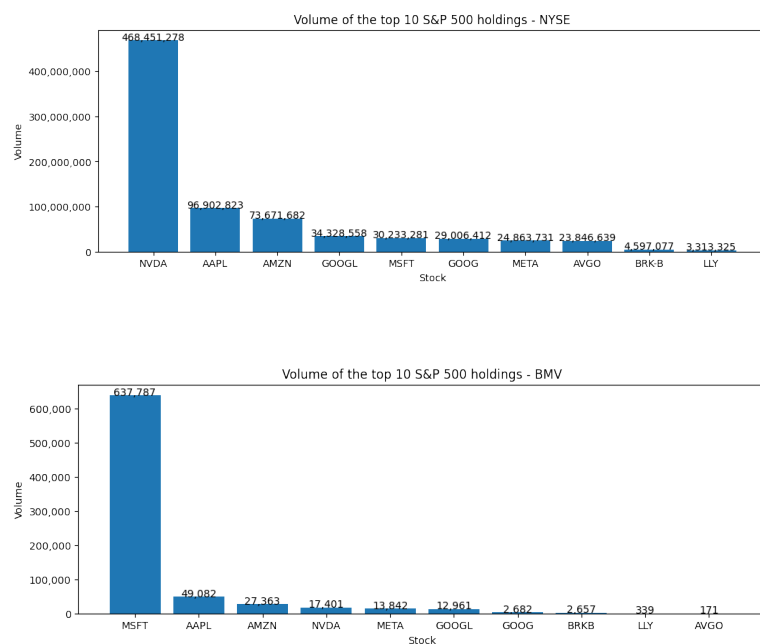
According to an article published in 2016, Mexico is the second most liquid and deep market in Latin America, only behind Brazil. However, both markets are significantly lagging in transaction volumes compared to the New York Stock Exchange⁵⁷. This situation forces participants in the Mexican Stock Exchange to pay special attention to the liquidity of a financial asset when investing.

⁵⁷ Jorge Orozco and David Ramírez. Análisis comparativo de los mercados bursátiles que integran el mila. *Revista Contexto*, Vol. 5, pp. 53-62, 2016

As mentioned previously, an asset that is difficult to transact can lead to significant liquidity losses, negatively affecting the investor's market performance. In extreme cases, this can result in sporadic trading of assets, meaning certain assets may not be traded on specific business days, making it very difficult for the manager to execute transactions.

In markets with these characteristics, investors generally tend to trade only those few assets with good liquidity metrics⁵⁸.

Currently, the lag in the BMV has persisted. We compared the average daily volume of the top ten weighted holdings in the SP 500 index, both on the New York Stock Exchange and the Mexican Stock Exchange. The results were as follows^{59,60}:



It can be observed that the trading volume is significantly lower in Mexico, even for the most prominent market stocks. This substantial disparity in volume between the two markets, with the U.S. market being considerably more liquid than the Mexican market, underscores the importance of liquidity risk when investing in this context. The higher trading volumes in the U.S. market enable investors to execute large transactions with minimal impact on asset prices, reflecting a more efficient market. In contrast, the lower trading volumes in Mexico indicate a less efficient market, where large transactions can significantly affect asset prices, leading to increased liquidity risk.

Consequently, portfolio managers must consider liquidity constraints when operating on the Mexican Stock Exchange due to this discrepancy in market activity levels. In less efficient markets, like Mexico, the cost of liquidity can be substantial, affecting the overall performance of an investment portfolio. Therefore, managing liquidity risk becomes critical for investors in the Mexican market to avoid potential losses and ensure optimal portfolio performance.

⁵⁸ Seth Armitage, Janusz Brzezczynski, and Anna Serdyuk. Liquidity measures and cost of trading in an illiquid market. *Journal of Emerging Markets*, Vol. 13, pp. 155-196, 2014

⁵⁹ Consulted on May 16, 2024

⁶⁰ SP Global. Sp500 constituents, 2024. URL <https://www.spglobal.com/spdji/en/indices/equity/sp-500>

Figure 2.5: Average trading volume of the top-weighted stocks in the S&P 500 on NYSE. Period: Jan'20 to Apr'24. Data retrieved from Yahoo Finance

Figure 2.6: Average trading volume of the top-weighted stocks in the S&P 500 on BMV. Period: Jan'20 to Apr'24. Data retrieved from EOD historical data

2.5 Assumptions

- Investors are rational.
- Only long positions are allowed in the portfolios, meaning short selling is not permitted.
- The returns of a financial asset follow a normal distribution with mean μ and variance σ^2 .
- The mean and standard deviation of daily returns are good measures to estimate the return and risk of an investment.
- The liquidity risk of a financial asset can be measured by the number of days it takes to unwind the position.
- The risk-free rate is constant.
- There is a fixed commission and tax for trading.
- No dividends are paid.

2.6 Hypothesis

Classic asset allocation models, such as the Sharpe ratio maximization, do not consider liquidity risk in their calculations, which can bias investors toward illiquid assets. Therefore, it is possible to enhance the results of this strategy in low-liquidity markets by incorporating a regularization factor that penalizes the weighting in assets with high liquidity risk.

2.7 Objectives

- Propose a new asset allocation strategy that considers liquidity risk, develop the analytical solution to this strategy, and discuss its properties.
- Propose an efficient frontier of portfolios that considers return, market risk, and liquidity risk.
- Compare the results of our method with the Sharpe ratio and verify that it penalizes illiquid assets.
- Simulate portfolios on the BMV, weighted by Sharpe ratio and the proposed strategy, with and without rebalancing, and analyze the results of both methodologies.
- Simulate portfolios weighted by Sharpe ratio on the NYSE and portfolios weighted by the proposed strategy on the BMV to analyze if it is preferable to pay an entry fee to operate in liquid markets.

3 Results

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The low liquidity of the Mexican stock market, compared to the United States market, prompts us to propose a method to solve the *portfolio problem* that considers not only market risk and the expected return of an investment but also liquidity risk.

In this chapter, we will develop a new method to address this problem. We will start by describing the methodology and the variables involved, followed by the mathematical formulation of the asset allocation model that includes liquidity risk. Additionally, we will analytically derive the model to find the optimal solution. Subsequently, we will propose a new efficient frontier of portfolios, presenting a Pareto front that incorporates the three previously mentioned factors.

Finally, we will implement the model in a simulated portfolio using numerical optimization techniques and then evaluate its performance against the traditional Sharpe method. This analysis will enable us to provide insights into the results obtained and discuss the practical implications of incorporating liquidity risk into portfolio management in markets with low liquidity.

3.1 Liquidity-Adjusted Sharpe Ratio

As stated in the previous chapter, the Sharpe Ratio (2.1.4) is a widely used metric that aids in maximizing the risk-return relationship of a portfolio. The following formula defines it:

$$SR = \frac{\mu_p - r_f}{\sigma_p} = \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}}$$

Where:

- μ_p : Portfolio return
- r_f : Risk-free rate
- σ_p : Portfolio volatility

The optimization problem is:

$$\begin{aligned} \max SR : & \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} \\ \text{subject to : } & \mathbb{1}^T w = 1 \end{aligned}$$

Note that this equation considers the expected return (numerator) and market risk (denominator). Given the objective of our work, we seek to add a factor that accounts for liquidity risk (2.3), which can be measured by the days it takes to liquidate a position in a financial asset:

$$N_{days} = \frac{C w \frac{1}{P}}{ADV}$$

Where:

- C : Capital or total portfolio value
- w : Asset weight in the portfolio
- P : Current asset price
- ADV : Average daily volume of the asset over the last three months

This formula only considers liquidity risk for a single financial asset; we can extend it to calculate the liquidity risk of a portfolio by summing the individual liquidity risks—considering that this risk is non-diversifiable—for the n assets it comprises:

$$\sum_{i=1}^n N_{days} = \sum_{i=1}^n \frac{C w_i \frac{1}{P_i}}{ADV_i}$$

Since in the Sharpe optimization problem, we seek to maximize the function; we can apply a regularization (2.2.1) by subtracting a liquidity factor from the function. This approach penalizes less liquid assets, reducing the sum of individual liquidity risks in the objective function. We will call this method "**Liquidity-Adjusted Sharpe Ratio**" (LA SR). Our objective function would be as follows:

$$LA\ SR = \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} - \lambda \sum_{i=1}^n \frac{C w_i \frac{1}{P_i}}{ADV_i}$$

Where λ is a positive hyperparameter characteristic of any regularization problem, in this context, it serves as a term that weighs the impact of liquidity risk on the Sharpe ratio, and we will call it the "*liquidity shrinkage factor*".

The previous function can be expressed in matrix form by making a change of variables. We consider $H = \left[\frac{1}{ADV_1 P_1} \quad \frac{1}{ADV_2 P_2} \quad \dots \quad \frac{1}{ADV_n P_n} \right]^T$, where both volume (ADV) and price (P) for each asset of the portfolio are known a priori by the manager. Since λ and the capital C are constants, we can define the following equation for the Liquidity-Adjusted Sharpe Ratio:

$$LA\ SR = \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} - \lambda C w^T H \quad (3.1)$$

With this formulation, we can pose our optimization problem as follows. It is worth noting that this function standardizes the weight vector w in the liquidity factor concerning the Sharpe function, which facilitates its derivation:

$$\begin{aligned} \max LA\ SR : & \quad \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} - \lambda C w^T H \\ \text{subject to : } & \quad w \mathbf{1}^T = 1 \end{aligned} \quad (3.2)$$

An essential feature of this method is that when liquidity risk tends to zero—which occurs in highly liquid markets or when the proportion of the portfolio's shares is insignificant relative to the ADV —the sum describing liquidity risk also tends to zero. This point implies that, in situations of low liquidity risk, our method converges to the classic Sharpe Ratio, as there is no need to incorporate liquidity constraints.

This formulation will help us find the set of optimal portfolios within the context of the Pareto front we are proposing based on the

liquidity shrinkage factor. Additionally, we propose a modification to the classic Sharpe Ratio problem that allows us to obtain the portfolio that maximizes the ratio between the Sharpe Ratio and liquidity risk without the need to establish a λ . We will call this the *Liquidity Variance Return Ratio (LVRR)*:

$$LVRR = \frac{\frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}}}{C w^T H} = \frac{SR}{C w^T H} \quad (3.3)$$

Drawing an analogy to the theory proposed by Markowitz¹, the proposed formulation of the Liquidity-Adjusted Sharpe Ratio represents a set of efficient portfolios in terms of Sharpe-Liquidity. These are those portfolios that, given a liquidity shrinkage factor, are efficient in terms of the proposed front, resulting in an efficient frontier. On the other hand, the LVRR represents a specific case of this frontier, where the highest ratio or the best relationship between the Sharpe Ratio and liquidity is obtained. The LVRR indicates how many units of the Sharpe Ratio we are obtaining per unit of liquidity risk assumed, and the larger it is, the better the Sharpe-Liquidity relationship.

¹ Harry Markowitz. Portfolio selection. *Journal of Finance*, Vol. 7, pp. 77-91, 1952

3.2 Analytical Optimization

Before analytically deriving the solution for the Liquidity-Adjusted Sharpe Ratio, we will simplify the analytical solution proposed by Merton² as described in the preliminaries of this thesis. To do this, we consider the objective function of the Sharpe Ratio:

$$\begin{aligned} \max SR : \quad & \frac{\mu_p - r_f}{\sigma_p} \\ \text{subject to :} \quad & \mathbf{1}^T w = 1 \end{aligned}$$

Assuming we have a risk-free asset within the portfolio, we can express the expected return of the portfolio as:

$$\mu_p = (w \mathbf{1}^T) \mu + (1 - w \mathbf{1}^T) r_f = w^T \mu + r_f - w^T r_f$$

Substituting the new expected value of the portfolio into the objective function, we obtain:

$$\frac{w^T \mu + r_f - w^T r_f - r_f}{\sigma_p} = \frac{w^T (\mu - r_f \mathbf{1})}{\sqrt{w^T \Sigma w}}$$

We can simplify this problem by regularizing the variance factor and subtracting the variance -or analogously the standard deviation- in the return maximization problem, leading us to:

² Robert Merton. An analytic derivation of the efficient portfolio frontier. *Cambridge University Press*, Vol. 7, pp. 1851-1872, 1972

$$\begin{aligned} \max SR : \quad & w^T(\mu - r_f\mathbb{1}) - \lambda(w^T\Sigma w) \\ \text{subject to : } & w\mathbb{1}^T = 1 \end{aligned}$$

Taking the derivative with respect to w , setting it to zero, and solving for w , we get:

$$\begin{aligned} \frac{\partial SR}{\partial w} &= \mu - r_f\mathbb{1} - 2\lambda\Sigma w = 0 \\ w &= \frac{1}{2\lambda}\Sigma^{-1}(\mu - r_f\mathbb{1}) \end{aligned}$$

With the equality constraint $w\mathbb{1}^T = 1$, we can find the value of λ :

$$\begin{aligned} \left(\frac{1}{2\lambda}\Sigma^{-1}(\mu - r_f\mathbb{1})\right)\mathbb{1}^T &= 1 \\ \mathbb{1}^T\Sigma^{-1}(\mu - r_f\mathbb{1}) &= 2\lambda \\ \lambda &= \frac{\mathbb{1}^T\Sigma^{-1}(\mu - r_f\mathbb{1})}{2} \end{aligned}$$

Substituting λ to find the optimal value of w :

$$\begin{aligned} w &= \frac{1}{2\lambda}\Sigma^{-1}(\mu - r_f\mathbb{1}) \\ w &= \frac{\Sigma^{-1}(\mu - r_f\mathbb{1})}{2\frac{\mathbb{1}^T\Sigma^{-1}(\mu - r_f\mathbb{1})}{2}} \\ w &= \frac{\Sigma^{-1}(\mu - r_f\mathbb{1})}{\mathbb{1}^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \end{aligned}$$

Bear in mind that this equation is the same as that found by Merton³, but the solution process was simplified by regularizing the function through the volatility term. Using the same logic, we can rewrite the optimization problem for the *Liquidity-Adjusted Sharpe Ratio*. We further assume, as in the previous exercise, that the portfolio includes a risk-free asset, which leave us with the following optimization problem:

$$\begin{aligned} \max LA SR : \quad & w^T(\mu - r_f\mathbb{1}) - \lambda_1(w^T\Sigma w) - \lambda_2(Cw^TH) \\ \text{subject to : } & w\mathbb{1}^T = 1 \end{aligned}$$

Taking the derivative, setting it to zero, and solving for w :

$$\frac{\partial SR}{\partial w} \text{ ApL} = \mu - r_f\mathbb{1} - 2\lambda_1\Sigma w - \lambda_2 C\mathbb{1}^T H$$

³ Robert Merton. An analytic derivation of the efficient portfolio frontier. *Cambridge University Press*, Vol. 7, pp. 1851-1872, 1972

$$w = \frac{1}{2\lambda_1} \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H)$$

We use the constraint $w \mathbf{1}^T = 1$ to find the value of λ_1 :

$$\begin{aligned} \frac{1}{2\lambda_1} \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H) \mathbf{1}^T &= 1 \\ \lambda_1 &= \frac{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H)}{2} \end{aligned}$$

Substituting in w :

$$\begin{aligned} w &= \frac{1}{2 \frac{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H)}{2}} \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H) \\ w &= \frac{\Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H)}{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda_2 C \mathbf{1}^T H)} \end{aligned}$$

Given that we consider two λ factors and have no additional constraints to find the value of λ_2 , we can assume $\lambda_2 = \lambda$, where λ is the previously defined liquidity shrinkage factor.

$$w = \frac{\Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda C \mathbf{1}^T H)}{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1} - \lambda C \mathbf{1}^T H)} \quad (3.4)$$

This formula allows us to determine the efficient weights in terms of the Liquidity-Adjusted Sharpe Ratio, given a value of λ , which can be interpreted as the level of penalty assigned to liquidity risk. The value of λ that maximizes the Sharpe-Liquidity Ratio represents the maximum Sharpe-Liquidity portfolio, previously defined as the Liquidity-Variance-Return Ratio. The LVRR can also be solved using numerical methods, with the SLSQP algorithm (2.2.4) being a viable alternative for this optimization. In addition, the Liquidity-Adjusted Sharpe Ratio (LA SR) can be addressed using SLSQP, combined with the TPE method (2.2.5) to identify the optimal value for λ .

3.3 Sharpe-Liquidity Efficient Frontier

With the optimization problem defined, we can establish a new efficient frontier. This frontier illustrates all possible combinations of an investment portfolio, showing the respective Sharpe Ratio (considering return and market risk) against the corresponding Liquidity Risk. This frontier follows the logic established by Markowitz but incorporates

liquidity risk, presenting another Pareto front. For a random portfolio composed of American Stocks trading in the Mexican Stock Exchange with a capital of \$1M MXN, we obtain the following frontier.

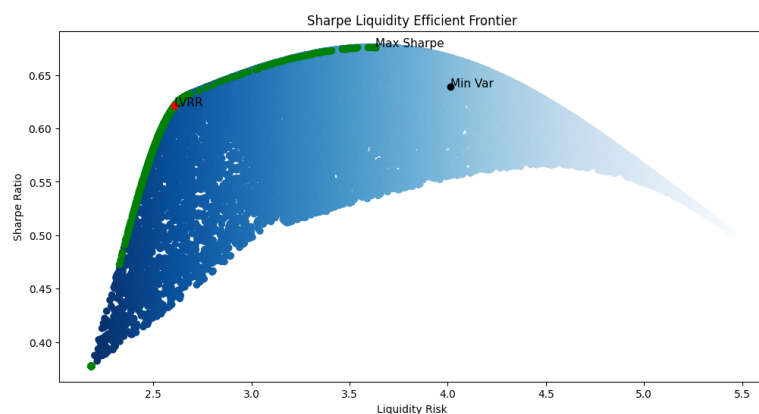


Figure 3.1: Sharpe-Liquidity Efficient Frontier for a portfolio with a capital of 1M MXN

In the shown frontier, it is important to highlight two key aspects. First, the green line represents all portfolios that are *optimal* in terms of Sharpe-Liquidity, meaning that for their given level of liquidity risk, no other portfolio can be found with a higher Sharpe Ratio. These portfolios were identified using the Liquidity-Adjusted Sharpe Ratio formula using different values of λ . Secondly, the point marked in red represents the LVR portfolio, which is the maximum portfolio in terms of Sharpe-Liquidity, as it maximizes this ratio.

We can note that liquidity risk significantly increases as we move towards the right side of the frontier, but the Sharpe Ratio also rises. This implies that seeking to raise a portfolio's Sharpe ratio incurs higher liquidity risk and vice versa. This behavior follows the logic of a Pareto Front, where improving one objective function necessitates sacrificing another. A similar phenomenon occurs in the Markowitz frontier but within the Mean-Variance context.

Adding to these observations, it is essential to emphasize that although the LVR portfolio maximizes the Sharpe-Liquidity ratio, it is not necessarily the best portfolio for all investors. The selection of a portfolio in this or any Pareto Front context should be based on the investor's individual characteristics and preferences. An investor may prefer to minimize liquidity risk or maximize the Sharpe Ratio, regardless of the outcome of the other function.

Another consideration regarding liquidity risk is that it is relative in nature. As previously mentioned, it is challenging to quantify due to its dependence on the average volume of the financial assets invested

in and the available investment capital. Therefore, the calculation of liquidity risk is multifactorial.

Also, is important to ponder that in scenarios where the invested capital is relatively small—such as a portfolio with a capital of \$10,000 MXN— or when the assets within the portfolio are highly liquid, as is often the case in the American market, the traditional Markowitz approach may be more appropriate than the method proposed in this thesis. In such cases, liquidity risk has a minimal impact on the portfolio's performance, making the simpler Markowitz framework more suitable for optimizing portfolios without the need for additional complexity.

To generalize this observation, we can conclude that the approach proposed in this thesis is not recommended as long as the shares bought with the invested capital are not significant concerning the liquidity characteristics of a set of financial assets. In such scenarios, liquidity risk is practically nonexistent, which does not justify the need for additional adjustments in the portfolio optimization model.

This commentary underscores the importance of considering specific market characteristics and invested capital when choosing a portfolio optimization approach. In highly liquid markets or when investing relatively small capital, the traditional Mean-Variance method is sufficient and efficient. However, in markets with lower liquidity or with larger investment capital, this thesis's approach, which incorporates liquidity risk, becomes a more appropriate tool for asset allocation.

3.4 *Implementation Mexican Stocks*

In this section, we will delve into the practical application of the method across two distinct scenarios: one involving Mexican stocks and another featuring American stocks that are traded on the Mexican Stock Exchange. It is important to note that in the second scenario, this stocks are made available to Mexican investors through the SIC⁴.

We'll begin with the analysis of Mexican stocks. For this, we gathered price and trading volume data for a selection of twenty Mexican equities, leveraging quotation data made available by the Mexican Stock Exchange. The dataset spans from January 1, 2020, to February 29, 2024, providing us with a robust timeframe that includes various market conditions, such as the post-pandemic recovery, inflationary pressures, and fluctuations in global markets.

⁴ The International Quotation System (SIC) is a platform that allows Mexican investors to trade stocks and Exchange Traded Funds (ETFs) listed on international markets, providing access to a wide range of global securities.

BMV. Mercado global, 2024. URL <https://www.bmv.com.mx/es/mercados/mercado-global>

These stocks were selected to represent a diverse range of sectors within the Mexican economy, ensuring a comprehensive view of market dynamics. The equities under consideration are:

| Ticker | Company |
|---------|--------------------------------|
| AC | Arca Continental |
| ALSEA | Alesea |
| ARA | Consortio ARA |
| BBVA | BBVA Bancomer |
| CUERVO | Casa Cuervo |
| ELEKTRA | Elektra |
| FRES | Fresco |
| GCC | GCC |
| GENTERA | Genera |
| GIGANTE | Grupo Gigante |
| GMXT | Grupo México |
| HCITY | Hoteles City Express |
| HERDEZ | Herdez |
| HOTEL | Grupo Hotelero Santa Fé |
| ORBIA | Orbia |
| PE&OLES | Peñoles |
| PINFRA | Grupo Pinfra |
| Q | Qualitas Controladora |
| VESTA | Corporación Inmobiliaria Vesta |
| VINTE | Vinte |

Table 3.1: Mexican Stocks Selected for Analysis: Tickers and Company Names

Next we can see a graphical representation depicting the evolution of closing prices, along with the corresponding average trading volumes throughout the specified period:

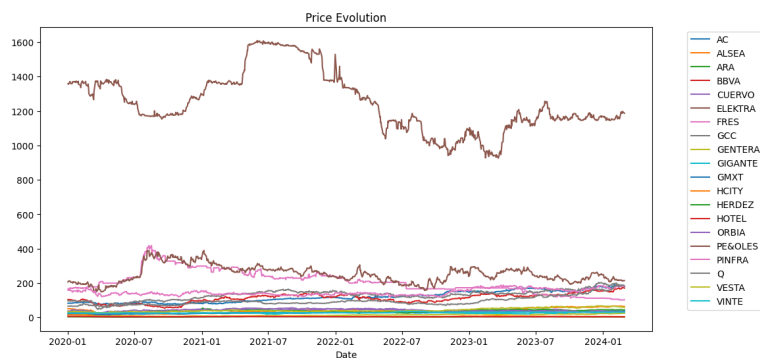


Figure 3.2: Mexican Stocks Price Evolution

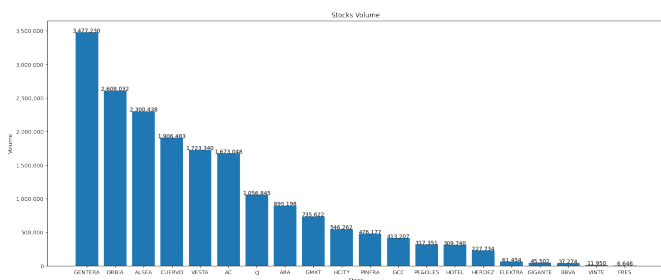


Figure 3.3: Mexican Stocks Average Daily Volume

It is worth noting that the stocks were selected to have varying trading volumes in order to illustrate the characteristics of our method. While the goal is not for the LVRR method to exclusively select stocks with the highest volumes, it is important that it identifies those stocks that have the best balance across the three analyzed dimensions: market risk, liquidity risk, and expected return. Additionally, it should uncover relationships between these factors, which would be challenging for the traditional Sharpe ratio method to detect.

Then again, the following subsections will focus on implementing the method in a portfolio composed of these stocks. It is important to highlight that this analysis aims solely to portray and comment on the method's advantages through a concrete example.

3.4.1 Asset Allocation

In order to optimize the portfolio, we'll assume a capital of \$1M MXN. This amount is sufficiently substantial to ensure significance when considering the number of shares purchased in relation to their average daily volume (ADV). To effectuate the optimization of the asset allocation models, we employed the SLSQP algorithm (2.2.4), we got the following results:

| Ticker | SR Weights (%) | LVRR Weights (%) |
|---------|----------------|------------------|
| AC | 18.1 | 19.5 |
| ALSEA | 0 | 0 |
| ARA | 0 | 0 |
| BBVA | 0.9 | 0 |
| CUERVO | 13.2 | 0 |
| ELEKTRA | 7 | 0 |
| FRES | 15.7 | 0 |
| GCC | 21.1 | 0 |
| GENTERA | 0 | 31.6 |
| GIGANTE | 0 | 0 |
| GMXT | 13.6 | 0 |
| HCITY | 0 | 0 |
| HERDEZ | 0 | 0 |
| HOTEL | 0 | 0 |
| ORBIA | 1.0 | 0 |
| PE&OLES | 0 | 0 |
| PINFRA | 0 | 0 |
| Q | 9.4 | 48.9 |
| VESTA | 0 | 0 |
| VINTE | 0 | 0 |

Table 3.2: Efficient weights in SR and LVRR for the portfolio composed of Mexican Stocks

As we can observe from the previous table, the LVRR method chooses to invest simply in AC, GENTERA, and Q, while the Sharpe method goes for AC, BBVA, CUERVO, FRES, GCC, GMXT, ORBIA, and Q. It is important to note that in a portfolio optimization problem the weights assigned to assets can sometimes be zero. This happens when the optimization algorithm identifies that certain assets do not contribute meaningfully to the desired risk-return or risk-return-liquidity profile. Specifically, an asset may be excluded if its inclusion

introduces excessive market risk, liquidity risk, or if it offers a relatively lower expected return compared to other available options.

An interesting point is that the proposed method not only selects the stocks with the highest volume in the portfolio, but also distributes it across different stocks. This means that it effectively considers not only liquidity risk but also takes into account the returns and market risk. We will complement this conclusion in the following section by examining the portfolio metrics, as well as the number of days it would take to unwind our position in the individual assets.

3.4.2 Backtesting and Performance Metrics

When performing the backtesting for the two optimized portfolios and comparing them against their respective benchmark, in this case, the S&P BMV IPC, we obtained the following results. It is important to mention that a risk-free rate of 5.5% was assumed, in order to keep the risk free constant throughout all the performed exercises, given that in the following sections we're testing against US stocks.

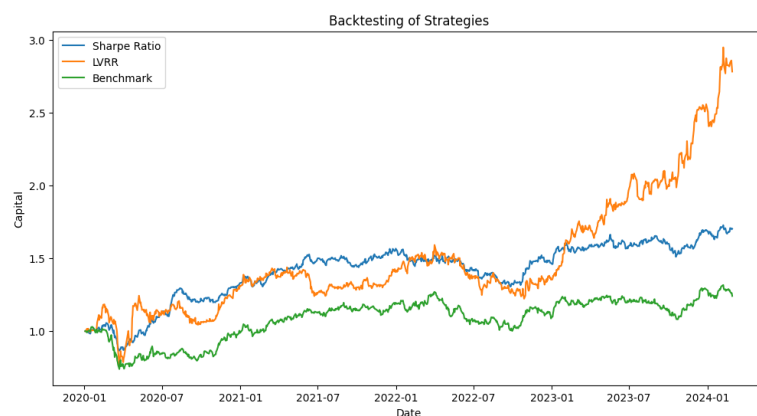


Figure 3.4: Backtesting for the efficient portfolios with Mexican stocks in SR and LVRR

| Metric | SR | LVRR |
|-------------------|-------------|-------------|
| Final Capital | 1,701,907.7 | 2,780,297.5 |
| Return | 13.6% | 24.5% |
| Volatility | 13.06% | 25.8% |
| Downside | 8.4% | 18.1% |
| Upside | 8.57% | 19.2% |
| Omega | 1.02 | 1.06 |
| Sortino | 0.9 | 1.05 |
| Days to Liquidate | 35 | 1 |

Table 3.3: Performance Metrics for the efficient portfolios with Mexican stocks in SR and LVRR

Our strategy consistently outperformed the traditional Sharpe ratio strategy across all metrics. Despite having higher volatility, our strategy achieved a higher Omega ratio, with an Omega of 1.06, compared to

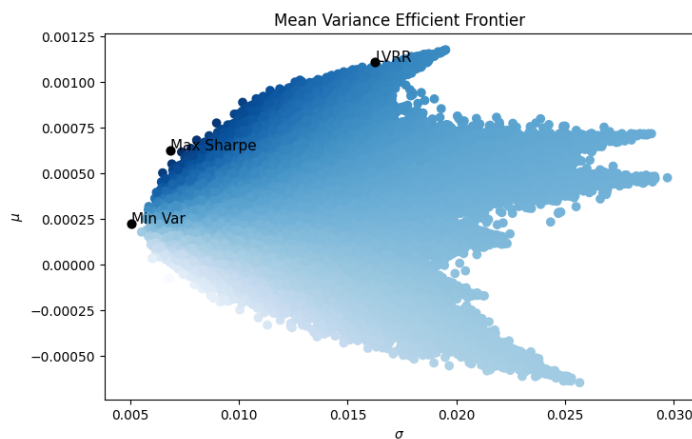
the 1.02 Omega of Sharpe strategy. Additionally, the Sortino ratio is 0.15 higher, with a Sortino of 1.05.

Although diversification may be a debatable issue—since our method selected only three stocks compared to the eight chosen by the Sharpe strategy—the results suggest that this did not have a negative impact in these specific case. The metrics show a better downside risk-return relationship. However, it's worth noting that while we reduce liquidity risk, we face an increase in market risk. This highlights the inherent trade-off between mitigating liquidity risk and taking on additional market risk.

The primary advantage of our method is evident in the final row of the table. While the Sharpe method would take approximately 35 days to fully liquidate our position, the LVRR method, due to its weightings, would require only a single day (results are rounded up since fractional days are not considered). Thus, the LVRR method successfully achieves its objective of reducing portfolio liquidity risk.

3.4.3 Efficient Frontiers

Now, we will plot the efficient frontiers, both in the context of Markowitz and as proposed in this thesis, to draw conclusions and make inferences about the results obtained. Below, we can observe the mean-variance efficient frontier.



We can observe that, in this case, our LVRR portfolio lies within the mean-variance efficient frontier, where it is evident that it shifts to the right within the frontier, accepting higher market risk in exchange for higher return. However, there may be instances where this portfolio may not fall within this efficient frontier, particularly when

Figure 3.5: Mean-Variance Efficient Frontier for the portfolios composed of Mexican Stocks

the portfolio manager preference is placed towards liquidity risk. This is not necessarily incorrect, as mentioned previously, because the efficient frontier we are proposing considers three dimensions. By only considering the mean and variance, we overlook the importance of liquidity risk. Next, we can observe the Sharpe-Liquidity efficient frontier.

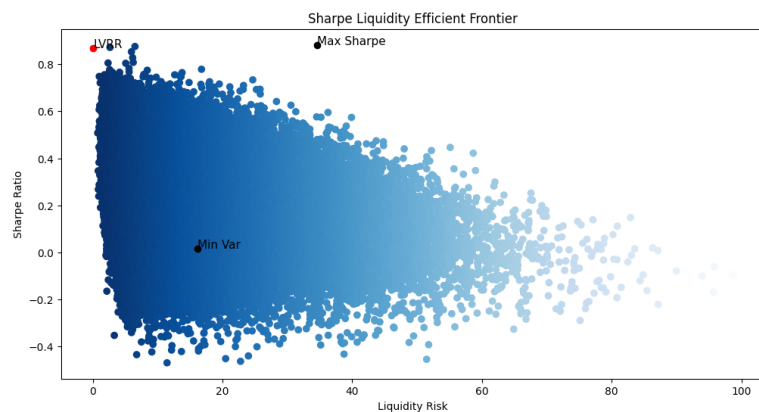


Figure 3.6: Sharpe-Liquidity Efficient Frontier for the portfolios composed of Mexican Stocks

The proposed portfolio lies in the upper left side of the frontier, with a liquidity risk of 1, meaning it would take one day to sell all the positions in our portfolio. In contrast, if we invest in the portfolio that maximizes the Sharpe ratio, we achieve a better relationship between market risk and return, but with a liquidity risk of nearly 36 days, which is how long it would take to sell all our positions. Selling a portfolio in 36 days poses a significant risk for the portfolio manager due to potential price fluctuations during that period, which could drastically affect the returns obtained. Moreover, managers often face liquidity constraints, meaning they cannot purchase assets that are so difficult to sell. Therefore, the LVRR is more viable in this aspect.

3.5 Implementation American Stocks

In this section, we will apply the previously discussed workflow to U.S. stocks listed on the Mexican Stock Exchange through the SIC. To carry out the implementation we collected the prices and trading volumes of the top ten stocks in the S&P 500⁵ using the quotation data available from the Mexican Stock Exchange. The data period spans from January 1, 2020, to February 29, 2024. The stocks are as follows:

⁵ Consulted on May 16, 2024.

SP Global. Sp500 constituents, 2024.
URL <https://www.spglobal.com/spdji/en/indices/equity/sp-500>

| Ticker | Company |
|--------|------------------------------|
| MSFT | Microsoft Corp |
| AAPL | Apple Inc. |
| NVDA | Nvidia Corp |
| AMZN | Amazon.com Inc |
| GOOGL | Alphabet Inc A |
| META | Meta Platforms, Inc. Class A |
| GOOG | Alphabet Inc C |
| BRK.B | Berkshire Hathaway B |
| LLY | Eli Lilly & Co |
| AVGO | Broadcom Inc |

Table 3.4: Companies with the highest weighting in the S&P 500 as of May'2024

Below is a graphical representation of the closing price evolution and their average volume during the mentioned period:

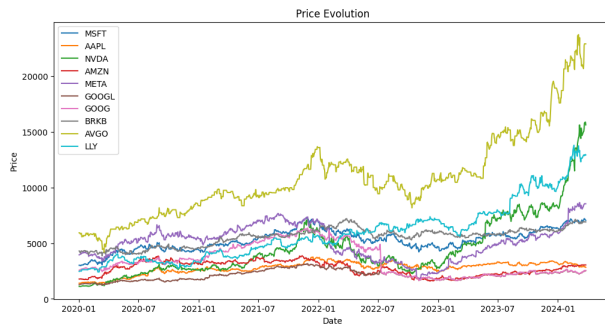


Figure 3.7: American Stocks Price Evolution

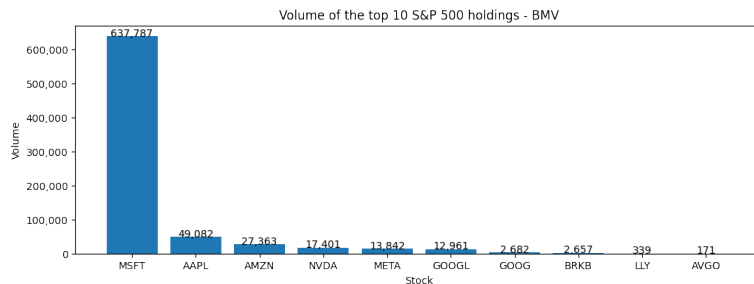


Figure 3.8: American Stocks Average Daily Volume

3.5.1 Asset Allocation

Next, to implement the Max Sharpe and LVRR strategies, we optimized the portfolio weights based on an initial investment of \$1M MXN. It is important to note that the effects of exchange rate fluctuations between the U.S. dollar and the Mexican peso (USD/MXN) were subtracted from this analysis, allowing us to focus solely on the variations in the stock prices themselves. The resulting optimized weights are as follows:

| Ticker | Company | SR Weights (%) | LVRW Weights (%) |
|--------|------------------------------|----------------|------------------|
| MSFT | Microsoft Corp | 0 | 23.9 |
| AAPL | Apple Inc. | 0 | 0 |
| NVDA | Nvidia Corp | 26.3 | 76.1 |
| AMZN | Amazon.com Inc | 12.4 | 0 |
| GOOGL | Alphabet Inc A | 0 | 0 |
| META | Meta Platforms, Inc. Class A | 0 | 0 |
| GOOG | Alphabet Inc C | 0 | 0 |
| BRK.B | Berkshire Hathaway B | 0 | 0 |
| LLY | Eli Lilly & Co | 39.0 | 0 |
| AVGO | Broadcom Inc | 22.3 | 0 |

Table 3-5: Efficient weights in SR and LVRW for the portfolio with the top ten stocks of the S&P500

As we can see, while the classical Sharpe method distributes investments among LLY, AVGO, AMZN, and NVDA, the proposed method opts to invest only in NVDA and MSFT. Note that the stocks of LLY and AVGO have the lowest trading volumes in the entire portfolio, with an average daily volume of 339 and 171, respectively, as shown in figure 4.6. Under the efficient weights of the Sharpe Ratio, a portfolio manager could take four days to liquidate their position, potentially resulting in significant liquidity losses.

In contrast, our method assigns a greater weight to two of the most liquid stocks in the portfolio, NVDA and MSFT. It is worth noting that the LVRW portfolio considers the individual liquidity of the stocks but does not exclusively favor the highest-volume stocks. Instead, it also takes into account risk and return. This is evident as the method recommends investing in NVDA despite it being only the fourth most liquid stock in the portfolio due to its favorable risk-return profile.

This approach provides a more nuanced perspective for the investor, allowing for a multifactorial analysis of both market and liquidity risk while simultaneously evaluating the expected return on investments. Although the diversification within this portfolio is debatable, this topic will be addressed in more detail after conducting a backtesting and analyzing the performance metrics of both portfolios. This analysis will be executed in the following section.

3.5.2 Backtesting and Performance Metrics

With the optimal weights from the Sharpe Ratio and LVRW, we conducted backtesting for both portfolios and compared them against a benchmark, which, given the nature of the stocks, is the S&P 500. The results showed that the LVRW portfolio had the highest return but also the highest volatility during the period.

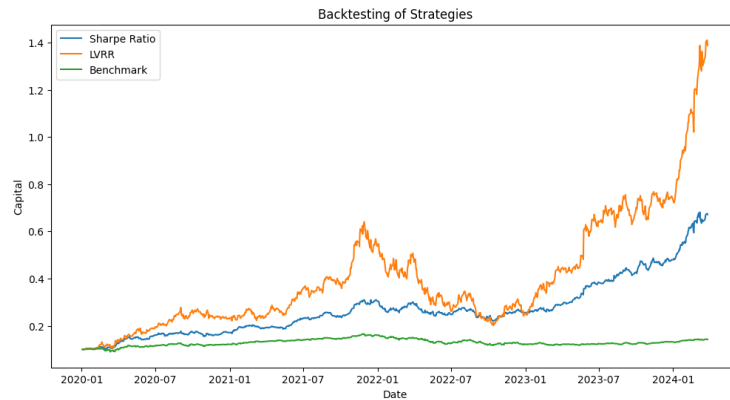


Figure 3.9: Backtesting for the efficient portfolios in SR and LVRR composed of American Stocks

In the following table we can see that the final capital and return are significantly higher with the proposed method. However, the volatility is nearly double compared to the Sharpe Ratio. This is primarily because the stock with the highest weight in the liquidity-adjusted portfolio is NVDA, which has shown both great return and high volatility in recent years. This fact confirms out the conclusion made in the previous section, reducing liquidity risk involves an exchange between liquidity risk and the Sharpe Ratio, where lowering liquidity risk could lead to higher volatility.

| Metric | SR | LVRR |
|-------------------|--------------|--------------|
| Final Capital | 6,714,492.35 | 9,603,861.46 |
| Return | 47.75% | 82.81% |
| Volatility | 22.98% | 43.15% |
| Downside | 12.7% | 21.1% |
| Upside | 19.3% | 35.5% |
| Omega | 1.51 | 1.68 |
| Sortino | 3.33 | 3.66 |
| Days to Liquidate | 4 | 1 |

Table 3.6: Performance Metrics for the efficient portfolios in SR and LVRR composed of American Stocks

We can also observe that our method achieved better performance metrics in terms of the relationship between downside risk and upside risk, as well as excess return and downside risk. With an Omega ratio of 1.68 compared to the Sharpe ratio's 1.51, and a Sortino ratio of 3.66 versus 3.33 for the traditional Sharpe method, the superiority of these metrics becomes clear.

The importance of these figures lies in the fact that, although we are accepting higher volatility, the growth potential —measured by upside risk- relative to the risk of loss -measured by downside risk— is significantly greater. This indicates that the accepted volatility does not entirely equate to loss risk. Instead, it reflects a greater dispersion toward gains rather than losses.

Also, liquidity risk is significantly reduced in the proposed method

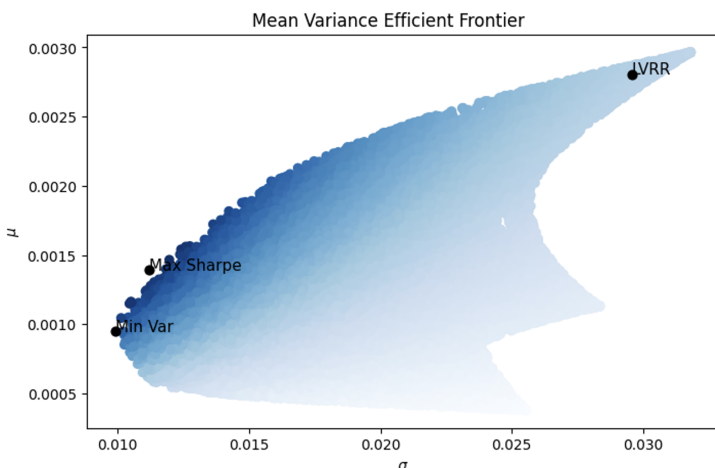
compared to the Sharpe Ratio, as shown in the last row of the performance metrics table (results are rounded up since there are no fractions of days). Which means that, in this case the LVRR method also achieves its goal of reducing portfolio liquidity risk.

In this scenario, an asset manager might opt to maintain their position in the Sharpe portfolio, which, despite generating lower returns, also exhibits lower volatility. However, it is crucial for them to recognize that fully liquidating the portfolio could take up to four days. As previously mentioned, managers often face liquidity constraints, requiring them to liquidate positions within a specific time frame.

From the results of both exercises, it is evident that the LVRR method can assist in reducing the portfolio liquidity risk to meet these constraints, potentially decreasing the liquidation period to as little as one day. This is achieved by optimizing the portfolio to reach the maximum Sharpe Ratio attainable given the level of liquidity risk.

3.5.3 Efficient Frontiers

Taking the same portfolio as before, along with the corresponding efficient portfolios for minimum variance, maximum Sharpe Ratio, and maximum Liquidity-Variance-Return Ratio (LVRR), we plotted the mean-variance efficient frontier for all possible combinations of these assets.



In the graph, it can be observed that the LVRR portfolio is outside the mean-variance efficient frontier proposed by Markowitz. This is primarily because we are considering liquidity risk in the calculation. Sometimes, our method may place greater weight on liquidity risk, which will bias this portfolio towards higher variance and/or lower

Figure 3.10: Mean-Variance Efficient Frontier for the portfolios composed of American Stocks

return, thus displacing it outside this efficient frontier. Then again, it is important to consider that the Pareto front proposed in this thesis includes mean, variance, and liquidity. Therefore, it may not be efficient in terms of mean and variance alone.

Nonetheless, like in the previous study case involving Mexican stocks, there may be cases where the portfolio will belong to the set of mean-variance efficient portfolios, particularly when the LVRR method emphasizes variance and mean. Also, when asset liquidity is exceptionally high or the available capital is relatively small, the LVRR portfolio will naturally converge with the Maximum Sharpe portfolio, situating it on the Mean-Variance frontier.

Next, we will plot the Sharpe-Liquidity frontier to verify that we are within the set of efficient portfolios on this front.

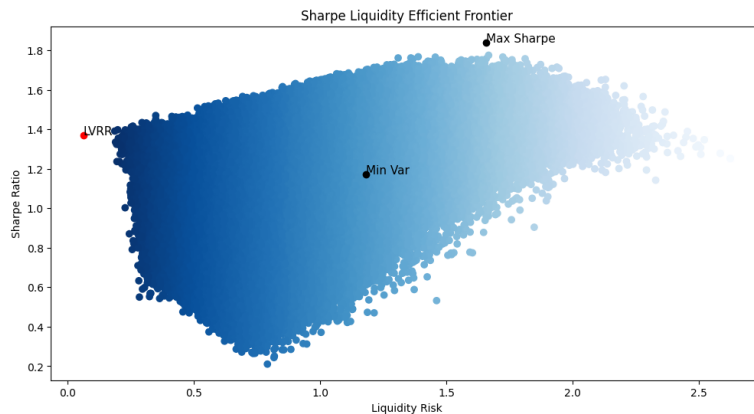


Figure 3.11: Sharpe-Liquidity Efficient Frontier for the portfolios composed of American Stocks

In this case, our portfolio is indeed on the Sharpe-Liquidity efficient frontier and presents the lowest liquidity risk, very close to zero, with the highest possible Sharpe Ratio. Therefore, this portfolio belongs to the set of optimal portfolios on this Pareto front. The image also shows that the minimum variance portfolio is not efficient in this context, similar to how the LVRR is not efficient in the Mean-Variance context. This clearly illustrates what we mentioned earlier about the Pareto front: optimality depends on the context and the front analyzed. An optimal portfolio in one context may not be optimal if considered outside that context.

Consequently, it is up to the portfolio manager to determine which terms are most important to optimize. They need to understand the environment in which they operate, as they should expect a different performance from a portfolio invested in the U.S. market compared to the Mexican market due to the inherent differences

between these markets. Therefore, they should apply the portfolio optimization method that best suits their objectives, preferences, and market characteristics.

By considering and prioritizing the most important metrics—or objective functions—the manager can select an appropriate asset allocation method that fits their needs and decide which is the "best" portfolio based on these. It is crucial to remember that to improve an objective function, such as liquidity, a penalty in the portfolio's performance in other aspects, such as volatility, must be accepted.

3.6 Considerations

As previously mentioned, the purpose of the two preceding sections was illustrative: to display, through two practical exercises, the advantages and disadvantages of the proposed method, along with important observations. The results do not imply that the method is inherently superior or inferior to its peers.

The first implementation illustrates a portfolio comprised of Mexican stocks, while the second features a portfolio that represents the American market, selecting the ten stocks with the highest weight in the S&P 500. The strong performance might be attributed to the specific behavior of the chosen stocks during the considered period.

Therefore, in the next section, we will delve deeper using simulations to analyze the behavior of the liquidity adjustment compared to the classic Sharpe Ratio. By using a broader sample of assets and considering various practical situations faced by a portfolio manager, we can determine whether the high performances observed in the exemplary portfolios in this section were a coincidence or if our method generally produces better results.

The findings presented in this section clearly illustrate the most important considerations of the method. Firstly, it was empirically and mathematically verified that the Liquidity-Adjusted Sharpe Ratio method and, consequently, the LVRR (Liquidity Variance Return Ratio) have a solution. This verification ensures that the proposed model is viable and can be applied in practice.

Secondly, a new efficient frontier was proposed based on the Pareto front analyzed in this research. This frontier considers three of the most critical factors in the investment portfolio environment: return, market risk, and liquidity risk. This novel efficient frontier, termed the

Sharpe-Liquidity frontier, enables portfolio managers to visualize and balance these three key factors, thereby facilitating investment decisions in a more holistic and market-conditioned manner.

Thirdly, it was found that, given the nature of a Pareto front, for an asset manager to improve the expected performance of their portfolio in one or more metrics, they must sacrifice performance in others. This phenomenon, known as a trade-off, is fundamental in multi-objective optimization theory and underscores the need to prioritize investment objectives according to the investor's preferences and the market's characteristics.

Furthermore, empirical analysis confirmed that liquidity risk is determined by two main factors: the market's liquidity and the number of shares constituting the portfolio. In instances where the number of shares purchased, given the portfolio value, is not significant relative to the average daily volume (ADV), liquidity constraints become unnecessary for portfolio optimization.

In these scenarios, the traditional Mean-Variance framework remains robust, offering a proven methodology for asset allocation. However, in less liquid markets, where liquidity risk is more pronounced, the LVRR method can achieve good results by effectively incorporating this additional risk factor. By accounting for liquidity risk, the LVRR method ensures a more tailored risk management strategy, thereby optimizing the overall performance of the portfolio.

Finally, these findings accentuate the importance of adapting asset allocation strategies to specific market conditions and the unique characteristics desired in an investment portfolio. Flexibility and knowledge about how and when to apply different optimization models allow asset managers to select the most appropriate approach according to their investment objectives, ultimately yielding better results. The continuous evaluation and adjustment of strategies in response to changing market conditions and portfolio dynamics are essential for sustained success in wealth management.

4 Application Cases

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To statistically evaluate the performance of the LVRR strategy compared to the Sharpe Ratio, we downloaded the historical prices and volumes of 140 U.S. companies traded on the BMV through the SIC, covering from January 1, 2020, to February 31, 2024. With this data, we will simulate portfolios with random asset selection across various application scenarios. In the first scenario, asset weights are optimized and invested over a specific period without any changes, resulting in a static backtesting. In the second scenario, the initial investment is made, and at each defined time horizon, the weights are re-optimized and adjusted, making it a dynamic backtesting through periodic rebalancing.

Finally, in the third scenario, we will obtain price and volume data from the NYSE for the same assets previously downloaded from the BMV. We will then compare the backtesting performance of the LVRR on the BMV with that of the Sharpe Ratio on the NYSE, factoring in a commission for trading in the U.S. market. This analysis will help us determine whether the benefits of the LVRR method outweigh those of the NYSE, considering the costs of operating in a more liquid market.

For each scenario, we will conduct 10,000 portfolio simulations. The goal of this experiment is to assess the performance of the LVRR across various portfolio compositions and to compare its outcomes against those achieved with the Sharpe Ratio. This will enable us to assess the behavior of the proposed methodology, identify patterns, and verify the consistency of the results. As a note, a fixed risk-free rate of 5.5%, a tax rate of 16%, and a broker fee of 0.125% on the operated amount were considered. Also, the USD/MXN variation was removed from the returns to eliminate any bias stemming from the exchange rate.

4.1 Static Backtesting of Portfolios

In this initial application case, utilizing the previously discussed information, the objective is to simulate the evolution of 10,000 distinct static investment portfolios with random asset selection. The workflow was developed as follows:

- 1.- Twenty stocks are randomly selected from the database.
- 2.- The weights are optimized for both the Sharpe Ratio and the LVRR, obtaining two efficient portfolios for each selection.
- 3.- Using the optimal weights, the evolution of both portfolios is simulated over the entire period (January 1, 2020, to February 31, 2024).
- 4.- Performance metrics, daily returns, and effective returns of each portfolio are obtained, calculating the mean, quartiles, and other relevant statistics.

This process was iteratively performed to simulate 10,000 different portfolios, each optimized using both methods. Upon comparing the results of both strategies, it was found that **the LVRR outperforms the classic Sharpe Ratio in 74.73% of the cases**. The distribution of daily returns for both strategies is illustrated in the following graph:

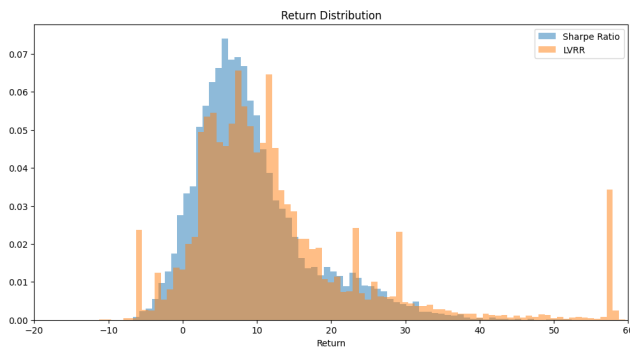


Figure 4.1: Daily Returns Histogram of Static Portfolios

We can observe that the daily returns of the Sharpe Ratio strategy are distributed approximately normally. At the same time, the LVRR strategy also follows this distribution but with lower kurtosis and higher positive skewness.

The distribution of the LVRR strategy exhibits heavier tails compared to the reference. This effect is mainly due to the liquidity adjustment, which generates a trade-off between objective functions, exchanging lower liquidity risk for higher market risk. Consequently, the results display a wider range due to greater volatility. To better visualize the results, a box plot was created with the information on the effective returns of the simulations of both strategies. It should be noted that outliers were removed from the graph.

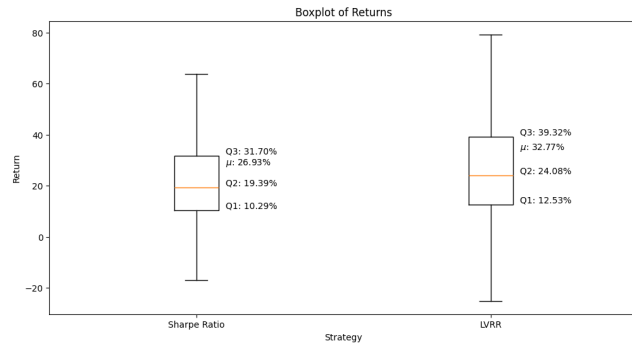


Figure 4.2: Effective Returns Box Plot of Static Portfolios

From this graph, we note that the expected effective return is 26.93% for the Sharpe Ratio and 32.77% for the LVRR, with medians of 19.39% and 24.08%, respectively. This indicates that the proposed strategy outperforms the Sharpe Ratio in both metrics. Additionally, the first and third quartiles are superior for the LVRR. Therefore, following a static investment strategy, that is, without considering periodic rebalancing, the LVRR achieves better results, with a 74% probability of outperforming the reference strategy in this sample.

Also, the proposed methodology surpasses the Sharpe ratio in terms of average performance metrics. The simulated strategy using the Sharpe ratio delivered average values of 3.70 for the Omega ratio and 1.12 for the Sortino ratio. In contrast, the LVRR achieved substantially higher averages, with 7.42 for the Omega ratio and 3.69 for the Sortino ratio. These findings suggest that our methodology offers a superior balance between return and downside risk, with a more favorable upside-downside risk profile compared to the Sharpe approach.

Regarding standard deviation, our original hypothesis that the proposed method increases the market risk—volatility—of a portfolio is confirmed. As shown in the histogram (Figure 4.1), the proposed method reduces liquidity risk at the cost of increasing market risk, as evidenced by the greater dispersion of the LVRR compared to the Sharpe Ratio.

Although the portfolio simulation in this case was performed using the same optimization data, with weights remaining static throughout the entire period, the results still offer valuable insights into the method's strengths and limitations. Moreover, these findings help to reinforce the conclusions drawn in the previous section. In the next part, the focus will be on eliminating this bias by optimizing the information prior to the backtesting data. This approach will enable the simulation of a more realistic scenario where an investor does not have foresight into the future performance of the assets in their portfolio.

4.2 Dynamic Backtesting of Portfolios

In this sub-section, we seek to perform the same experiment as in the previous part but consider periodic changes in the structure of the portfolios. We will achieve this by dynamically rebalancing the weights. Let t_0 be the initial point in time for a dataset, t_1 the subsequent period given a window established by the manager, t_2 two windows later, and so on until the last window t_n . The workflow for dynamic portfolio backtesting will proceed as follows:

- 1.- Randomly select twenty stocks from the database.
- 2.- Define the annual periodicity n for rebalancing. If defined quarterly, there will be four rebalancing events per year; semi-annually, there will be two per year; and annually, there will be one.
- 3.- Partition the database according to the n time windows and the m years of available information. This means the historical data will be divided into $m \times n$ windows.
- 4.- Optimize the weights using the first time window t_0 and simulate the portfolio performance for the subsequent period, i.e., from t_1 to t_2 . Once this window ends, optimize the weights again using data from t_1 to t_2 and simulate from t_2 to t_3 . Repeat this process iteratively until the simulation for the final period, t_{n-1} to t_n .
5. Using the portfolio's historical evolution, obtain the performance metrics and the effective and daily returns and calculate the mean, quartiles, and other relevant statistics.

The graphical representation of this backtesting methodology is illustrated below.

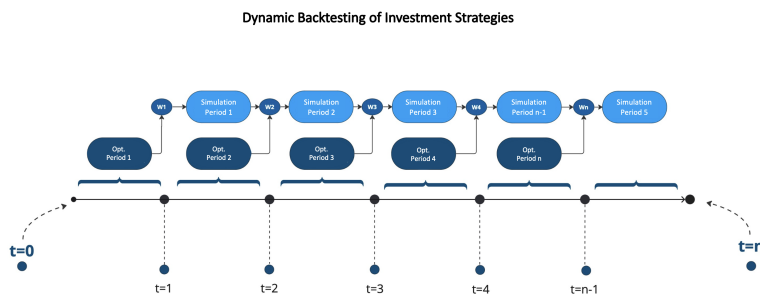


Figure 4.3: Backtesting Schematic for a Dynamic Portfolio

This process was carried out for both the Sharpe Ratio strategy and the liquidity-adjusted strategy for 10,000 different portfolios, optimizing the rebalancing window (considering periods of three, six, or twelve months). Given the logic of this simulation, there is no information on portfolio performance within the optimization data, making this methodology more realistic for an asset manager. The results will help

us better evaluate the strategies' performance. With the backtesting results, we found that **LVR** achieves **higher returns than the classic Sharpe Ratio in 80.79% of cases**. The histogram of the daily returns for both portfolios is shown below.

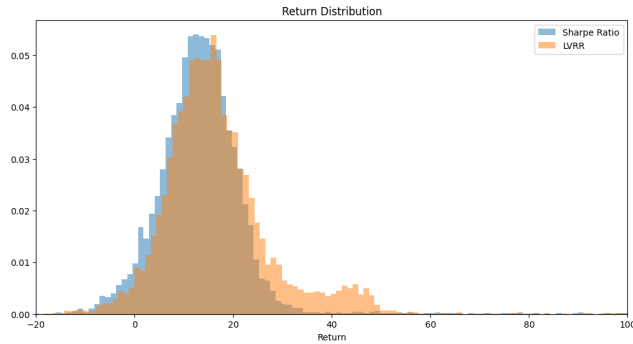


Figure 4.4: Daily Returns Histogram of Dynamic Portfolios

In this figure, we observe that periodic rebalancing shifts the return distribution of the LVR to the right compared to the Sharpe strategy, increasing the positive skewness of the histogram. This shift is favorable, as a greater positive skewness indicates a higher likelihood of gains or positive returns. Furthermore, the previous histogram shows that the proposed model presents a higher expected value than the classic Sharpe Ratio, evidenced by the LVR histogram's shift to the right. Additionally, this strategy appears to have a higher number of outliers in the gains region. This can be more clearly seen in the following effective returns boxplot.

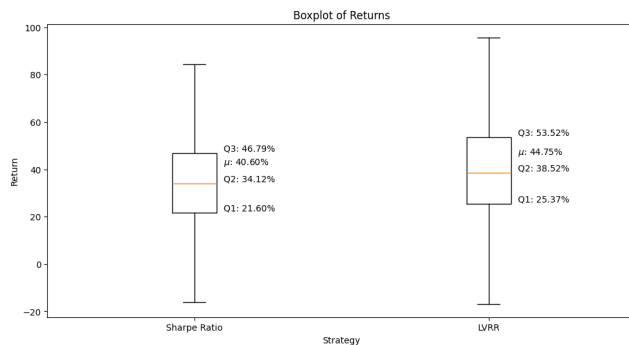


Figure 4.5: Effective Returns Box Plot of Dynamic Portfolios

The Sharpe Ratio strategy shows an average of 40.60% and a median of 34.12% in its effective returns for the various simulated portfolios. In comparison, the LVR presents an average of 44.75% and a median of 38.52%. This confirms that the proposed strategy exhibits greater positive skewness compared to the classic Sharpe approach. In general, performing periodic rebalancing adds more value to the investor, increasing the likelihood of obtaining profits.

Moreover, although the LVRR method shows higher volatility compared to the classic Sharpe Ratio, this is not necessarily negative. Measuring volatility as the dispersion of returns from the mean does not consider that positive dispersion is favorable, as it indicates a higher probability of gains. In figure 4.5, we can see that the positive dispersion in the proposed method is significantly higher than the negative dispersion. This means that although the market risk of the portfolio increases, this increased risk is more skewed towards gains rather than losses.

Separating market risk into two components, known as upside and downside risk, offers a better perspective on risk management. When holding a long position in a portfolio, upside risk refers to the potential for positive returns or gains, while downside risk pertains to negative returns or losses. This distinction allows investors to differentiate between beneficial and detrimental volatility within their portfolios. By carefully selecting assets that individually exhibit greater upside risk compared to downside risk, the portfolio as a whole can be structured to have a positive bias, making it more likely to experience gains rather than losses. The findings of this experiment suggest that the LVRR helps maintain a positive skew in portfolio returns. However, it is important to note that this relationship hinges on the individual attributes of the assets, which states the critical role of asset selection in the investment process. A strong asset allocation strategy is fundamentally built on rational asset selection.

To further support this conclusion, we calculated the Omega and Sortino ratios, both of which factor in downside risk. The Sharpe Ratio strategy produced an average Omega of 7.87 and a Sortino ratio of 1.12. In comparison, the LVRR strategy exhibited markedly higher values, with an average Omega of 13.61 and a Sortino ratio of 8.07. These metrics strongly validate our findings. First, portfolio rebalancing consistently enhances the performance of any asset allocation strategy when compared to static approaches. Furthermore, our methodology not only yielded superior performance metrics but also demonstrated a more favorable risk profile. With an Omega considerably greater than one, the additional risk was skewed toward upside potential.

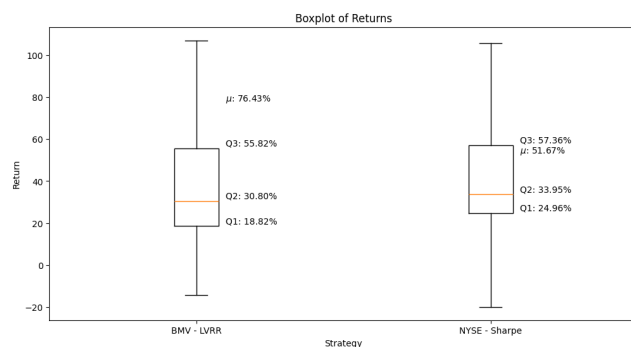
The results of the backtesting simulations suggest that, at least in the Mexican Stock Exchange, the LVRR consistently outperforms the Sharpe Ratio across a significant number of cases, regardless of whether a static or dynamic strategy is employed. We attribute this outcome to the integration of liquidity risk into the objective functions, which synergizes with market risk and expected return, resulting in a more effective asset allocation in low-liquidity markets.

4.3 Comparison of LVRR and SR Across Markets

In this final application case, we incorporated the closing prices and volume data from the New York Stock Exchange, with prices converted to Mexican pesos for the same 140 companies previously downloaded. As noted earlier, the USD/MXN variation was removed to eliminate any potential bias related to exchange rate fluctuations. The objective is to compare the results of the Sharpe Ratio method implemented on the New York Stock Exchange (where the volume is significantly higher) against the LVRR on the Mexican Stock Exchange. The backtesting approach was as follows:

- 1.- From the database containing information on the Mexican Stock Exchange, twenty stocks are randomly selected.
- 2.- From the second dataset, containing information on the American stock exchange, the same twenty stocks from the previous step are taken.
- 3.- The weights are optimized using the LVRR for the portfolio trading on the Mexican Stock Exchange and the Sharpe Ratio for the portfolio trading on the New York Stock Exchange.
- 4.- For each dataset, the portfolio evolution is simulated, and the corresponding performance metrics are obtained. We assumed an extra commission of 30 dollars (approximately 600 Mexican pesos at the current exchange rate) per purchase order in the American market.

Analyzing the results for the simulated portfolios, it was found that **only 43.48% of the portfolios optimized by the LVRR on the BMV achieve better results than those optimized by the Sharpe Ratio on the New York Stock Exchange** in terms of effective returns. A summary of the results is presented below.



In both cases, the mean is highly skewed relative to its median, especially in the backtesting results for the Mexican Stock Exchange portfolios, which have an average return of 76.43% and a median of 30.80%. On the other hand, on the New York Stock Exchange, the

Figure 4.6: Boxplot of Effective Returns from Asset Allocation Methods Applied to NYSE and BMV Portfolios

average is 51.67% compared to a median of 33.95%. Because of this, it is more appropriate to use medians to compare the results, where the Sharpe Ratio strategy on the New York Stock Exchange showed better results.

This, coupled with the fact that in only 43.48% of the cases, the proposed method achieved better results, leads us to conclude that the Sharpe Ratio, when implemented in highly liquid markets¹, is clearly superior to the LVRR method applied in low liquidity markets, which means that, for Mexican investors is preferable to trade on the NYSE rather than the BMV, where the benefits of higher liquidity outweigh the potential advantages of the LVRR approach.

While this is an important point to consider, it is not a disadvantage per se of the method, as investors will always lean towards trading in highly liquid markets². Such markets allow investors to buy and sell assets more efficiently, practically eliminating liquidity risk due to the intrinsic nature of the market. The keyword here is "efficiently", given that liquidity is directly tied to market efficiency. So, considering the NYSE is the largest and most active market globally, it's understandable that the Sharpe Ratio performs better in such an environment.

An efficient market provides several advantages for investors. For example, in this setting, the active trading of most assets boosts liquidity and minimizes trading friction, making it easier for investors to unwind positions and rebalance portfolios with minimal impact on the assets' spot prices. Additionally, heightened competition among brokers in efficient markets typically results in lower commission fees, further reducing transaction costs.

However, for an investor in Mexico, accessing the American market can be challenging due to various restrictions, such as the minimum amount needed to operate, additional commissions, and the need for an authorized intermediary. These barriers can limit Mexican investors' ability to capitalize on opportunities in more efficient markets like the United States.

Ultimately, while it is always preferable to trade in highly liquid markets, the LVRR method presents itself as a novel alternative for those investors who only have access to low-liquidity markets. This approach addresses the portfolio optimization problem with enhanced precision in these environments by incorporating the most relevant factors that investors must consider—expected return, market risk, and liquidity risk—when operating in this context.

¹ It is important to mention that the classic Sharpe Ratio was developed in this context

² D. L. Hanson and C.F. Menezes. Risk aversion and bidding theory. *University of Kansas Press*, 1968

5 Conclusions

The primary objective of this research was to develop an asset allocation method specifically designed for portfolio optimization in low-liquidity markets. To achieve this, we introduced a liquidity regularization term into the Sharpe Ratio, resulting in the creation of the Liquidity-Adjusted Sharpe Ratio. This method extends beyond merely considering market risk and portfolio returns; it also integrates the liquidity risk inherent in the financial assets. By optimizing this function, we reduce the weighting on less liquid assets, harmonizing the relationship between mean, variance, and liquidity risk. Additionally, we proposed a variant of this approach, known as the LVRR, which maximizes the relation between the Sharpe Ratio and liquidity risk.

The analysis conducted in this study demonstrated that the proposed method can be solved both analytically and through numerical optimization techniques. It was verified that, in theory and empirical practice, the method is effective. This means it can be applied by any investor interested in improving their portfolio management in low-liquidity markets. Notably, this method not only complements but also expands the classical portfolio optimization theory proposed by Markowitz and Sharpe.

Incorporating liquidity risk into the analysis adds a new dimension to investment decision-making, reflecting the complex realities of financial markets. In this sense, the proposed method aligns with the principles established by Markowitz and Sharpe and offers an evolution of these concepts, adapting them to investor's needs. Also, the simplicity with which this method can be implemented is a crucial aspect. Despite the underlying complexity of considering liquidity risk, the model's structure allows for straightforward application.

In some cases, this portfolio may not be optimal on the Mean-Variance front, mainly because the preference for penalizing liquidity can shift it away from the optimal set of portfolios. Therefore, we proposed a new Pareto front named the Sharpe-Liquidity Frontier that considers the three factors optimized in the problem.

Within this new frontier, both the Liquidity-Adjusted Sharpe Ratio for any shrinkage factor and the LVRR portfolio are considered optimal. This new Pareto front offers a more comprehensive visualization of optimality for any investor considering liquidity risk in their portfolio construction. The Sharpe-Liquidity frontier follows the logic of any other Pareto front, where improving one objective function necessarily involves a trade-off in another.

Through the analysis of the method using different backtesting methodologies, it was discovered that this trade-off usually manifests between liquidity risk and volatility (market risk). This is visualized in the higher volatility of the LVRR method, whose simulation results show heavier tails, lower kurtosis, and, at the same time, greater positive skewness.

The identification of positive skewness in the histogram of backtesting returns indicated that an increase in volatility is not inherently negative. In most cases, the portfolio's upside risk outweighs the downside risk, as evidenced by the higher Sortino and Omega ratios achieved by the proposed method. This demonstrates that the strategy effectively capitalizes on volatility, amplifying potential gains while managing downside exposure. This relationship could be maintained if financial assets with greater upside dispersion than downside are selected, calculating the standard deviation for both cases. This decision is at the discretion of the manager, who could significantly improve the proposed method's results by making a good asset selection.

Backtesting also revealed that for static and dynamic portfolios, the LVRR achieves better results than the Markowitz method in approximately 70-80% of cases, at least in the Mexican Stock Exchange. This demonstrates that liquidity provides valuable information that, when combined with risk and return information, generates synergy.

In dynamic backtesting simulations, it was observed that periodically rebalancing a portfolio increases the probability of gains. In these simulations, asset selection was random, so results are expected to improve if the investor conducts a thorough analysis of the economic cycle and context, as well as the momentum and fundamentals of the assets in which they will invest. This highlights the importance of correct asset selection, emphasizing that proper analysis generally leads to good investment results and that combining a quantitative asset allocation technique with qualitative analysis improves the performance of an investment portfolio.

On the other hand, comparing the proposed method in low-liquidity markets with the Sharpe Ratio on the New York Stock Exchange showed that the latter option generated better results. Therefore, if an investor has the opportunity to invest in more liquid markets, it is highly recommended to take advantage of it.

Nonetheless, the method shows good results in low-liquidity contexts. For investors who do not have access to developed markets and need to operate in shallow markets, it is recommended to consider liquidity risk when allocating capital within a set of assets. This not only reduces this type of risk but also incorporates valuable information about the financial assets that make up the portfolio and how they will behave in the future, giving a better asset allocation approach when investing in this kind of market.

A crucial consideration of the proposed method is that in highly liquid markets, the LVRR's regularization factor tends to diminish, rendering it almost negligible. As a result, the outcomes produced by the LVRR in these environments will be virtually identical to those of the traditional Sharpe Ratio.

Given this, we strongly recommend defaulting to the classical Sharpe Ratio in such scenarios, as utilizing the LVRR would only lead to unnecessary computational overhead and increased processing time without yielding any significant advantage. This reinforces the idea that the LVRR is a particular method for scenarios where liquidity risk is a prominent concern, and in highly liquid markets, the Sharpe Ratio remains the optimal choice.

Therefore, it is emphasized that a portfolio manager should select the portfolio optimization method that best suits them based on the investor's objectives and the market characteristics in which they operate. As demonstrated throughout this thesis, the performance of applied methods completely depends on the context.

Finally, we conclude that the LVRR enhances portfolio performance in low-liquidity markets, positioning itself as a viable alternative for portfolio managers operating in these environments. Although liquidity risk is inherently subjective, the proposed method addresses this by quantifying liquidity risk and integrating it into the asset allocation process. This approach not only reduces the bias linked to inaccurate valuation but also provides a more objective and robust framework for managing portfolios in markets with liquidity constraints.

5.1 *Future Work*

Although the objectives of this work were fully and effectively achieved, there are still some areas that could further complement the research. Additionally, the vast field of portfolio optimization, beyond just considering asset or portfolio returns and market risk, offers several avenues for further exploration. Below, we highlight some of these potential areas for future research:

- Develop, apply, and implement the liquidity regularization term in other asset allocation methods beyond Sharpe, such as the Minimum Variance, Target Minimum Semivariance, Black-Litterman methods, among others.
- Implement the LVRR method with a focus on Mexican stocks, applying the same portfolio backtesting methodology used in this study to analyze the distinctive market dynamics of Mexico's stocks landscape.
- Apply the LVRR method within the context of Mexican AFOREs, as these institutions are required to invest in Mexican stocks despite the low liquidity of the market. It would be important to incorporate these regulatory constraints into the optimization problem.
- Expand the LVRR method to portfolios that include a broader range of financial assets beyond equities, such as fixed income securities, derivatives, and FOREX.
- Implement the LVRR method in other low-liquidity markets to assess its effectiveness in different market environments.
- Introduce additional risk factors as part of the regularization process, such as credit risk and operational risk, to capture a broader risk profile in the optimization.
- Use the liquidity risk measure developed in this work as a performance metric for investment portfolios, aiming to quantify how changes in liquidity risk affect both returns and market risk.

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