A Second Order Sliding Mode Controller with Predefined-Time Convergence

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***Abstract*—This paper presents the basis to design a well-suited control law which guarantees predefined-time convergence for a class of second-order systems. In contrast to the case of finite-time and fixed-time controllers, a predefined-time controller allows to set the bound of the convergence time, explicitly during the control design. Furthermore, in the case of no disturbance, the least upper bound of the convergence time can be predefined directly from the control definition. A Lyapunov-like characterization for predefined-time stability is performed. Numerical results are discussed to show the reliability of the proposed method.**

***Index Terms*—Predefined-Time Convergence; Second-order Sliding Mode Control; Lyapunov method**

I. Introduction

Most applications of dynamical system design require to meet some performance constraints. For the case of control, observation and estimation, those requirements are usually related to fast responses while being robust to uncertainties, such as external disturbances or parameter variations. For those cases, sliding mode algorithms have been one of the most promising methods [1], [2].

A primary feature of the sliding mode control is the finite time stability [3]–[5]. However, the stabilization time is often an unbounded function of the initial conditions of the system. To overcome this drawback, making settling time bounded for any initial condition, a stronger form of stability, called fixed-time stability, was introduced by [6] for homogeneous systems and by [7]–[9] for systems with sliding modes. The settling time of fixed-time stable systems presents a class of uniformity to their initial conditions. The references [10]–[14] analyze a class of systems where an upper bound of the fixed stabilization time is a tunable parameter. This structural advantage allows coping with the problems related to the estimation of the convergence time.

Recent efforts to design predefined time controllers for high order systems are exposed in [15], [16]. However, the presence of singularities in the closed-loop dynamics makes the use of this controller restricted to particular cases. To improve the applicability of this class of second-order controllers, the reference [17] presents a variable structure controller which switch between different operation regimes, avoiding possible singularities.

This paper presents a novel sliding mode controller with predefined-time convergence. In contrast to the mentioned methods, the current proposal allows designing a nonsingular controller without the use of switching between regimes. With this aim, firstly, it is presented a generalized Lyapunov condition for predefined time stability. Secondly, with these stability conditions, it is designed a second-order sliding mode controller. Finally, through all the paper, some rigorous proofs are presented for all the proposed results.

The outline of the paper is as follows. In Section II, some basics on predefined-time stability and gamma function are recalled. The main results concerning the Lyapunov characterization of predefined-time stability are given in Section III. Based on this characterization, a robust controller is derived for second-order systems with bounded matched perturbation. Numerical simulations are presented in Section IV to illustrate the effectiveness of the proposed controller.

II. Preliminaries

A. On predefined-time stability

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{\rho}), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state. The vector $\boldsymbol{\rho} \in \mathbb{R}^b$ stands for the parameters of system (1), which are assumed to be constant, i.e., $\dot{\boldsymbol{\rho}} = 0$. Furthermore, there is no limit in the number of parameters, so b can take any value in the set of natural numbers $\mathbb{N}_0$. The function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear, and the origin is assumed to be an equilibrium point of system (1), so $\boldsymbol{f}(\boldsymbol{0};\boldsymbol{\rho}) = \boldsymbol{0}$. The initial conditions of this system are $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \mathbb{R}^n$.

Definition II.1 (See [4])**.** The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $\boldsymbol{x}(t,\boldsymbol{x}_0)$ of (1) reaches the equilibrium point at some finite time moment, i.e., $\forall t \geq T(\boldsymbol{x}_0) : \boldsymbol{x}(t,\boldsymbol{x}_0) = \boldsymbol{0}$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the *settling-time function*.

Definition II.2 (See [8])**.** The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, i.e. $\exists T_{\max} > 0 : \forall \boldsymbol{x}_0 \in \mathbb{R}^n,\ T(\boldsymbol{x}_0) \leq T_{\max}$.

Remark 1. Assuming that the origin of (1) is fixed-time stable, the bound $T_{\max}$ in Definition II.2 is trivially non-unique; for instance, note that $T(\boldsymbol{x}_0) \leq \lambda T_{\max}$ with $\lambda \geq 1$. This motivates the definition of a set which contains all the bounds of the settling-time function.

Definition II.3 (See [10])**.** Let the origin of system (1) be fixed-time-stable. The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \left\{ T_{\max} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\max},\ \forall\, \boldsymbol{x}_0 \in \mathbb{R}^n \right\}.$$

Remark 2. For some applications such as state estimation, dynamic optimization, fault detection, among others, it would be convenient that the trajectories of system (1) reach the origin within a time $T_c \in \mathcal{T}$, which can be defined in advance as a function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. This is the main idea of predefined-time-stable systems.

Definition II.4 (See [14])**.** For the system (1) parameters $\boldsymbol{\rho}$ and a constant $T_c := T_c(\boldsymbol{\rho}) > 0$, the origin of (1) is said to be *predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

If this is the case, T_c is called a *predefined-time*.

Remark 3. It would be desirable to choose $T_c = T_c(\boldsymbol{\rho})$ not only as a bound of the settling-time function $T_c \in \mathcal{T}$, but as the least upper bound, i.e., $T_c = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0)$. However, this selection requires complete knowledge about the system, compromising its application to uncertain systems.

B. On the incomplete gamma function inverse

Recall the definition of the *gamma* function:

Definition II.5 (See [18], [19])**.** Let $a > 0$. The *gamma* function is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) \mathrm{d}t. \tag{2}$$

Splitting the integral (2) at a point $x \geq 0$, two incomplete gamma functions are obtained. This motivates the following definitions.

Definition II.6 (See [18], [19])**.** Let $a > 0$ and $x \geq 0$. The *incomplete gamma function* is defined as

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t) \mathrm{d}t.$$

Definition II.7 (See [18], [19])**.** Let $a > 0$ and $x \geq 0$. The *regularized incomplete gamma function* is defined as

$$Q(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)}.$$

Definition II.8. Let $a > 0$ and $x \geq 0$. The *regularized incomplete gamma function inverse* $Q^{-1}(a, \cdot) : (0, 1] \to [0, \infty)$, is defined as the unique function satisfying $Q^{-1}(a, Q(a, x)) = x$.

Remark 4. Note that $Q(1, x) = \exp(-x)$, $Q(a, 0) = 1$, and $Q(a, x) \to 0$ as $x \to \infty$. Consequently, from Definition II.8, $Q^{-1}(a, 1) = 0$.

III. Main Results

A. A generalized Lyapunov characterization of predefined-time stability

The following theorem presents a Lyapunov characterization of predefined-time stable systems.

Theorem III.1. *If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $V(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$,
(ii) $V(\boldsymbol{x}) \geq 0$ and,
(iii) any solution $\boldsymbol{x}(t)$ of (1) satisfies

$$\dot{V}(\boldsymbol{x}) \leq -\frac{\alpha^{\frac{\beta q - 1}{p}} \Gamma\left(\frac{1-\beta q}{p}\right)}{p T_c} \exp\left(\alpha V(\boldsymbol{x})^p\right) V(\boldsymbol{x})^{\beta q} \tag{3}$$

for $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and constants $T_c := T_c(\boldsymbol{\rho}) > 0$, $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$ such that $\beta q < 1$.

Then, the origin of system (1) is predefined-time stable with predefined time equal to T_c.

Proof. From the differential inequality (3), $V(\boldsymbol{x}(t))$ satisfies $V(\boldsymbol{x}(t)) \leq \left[\frac{1}{\alpha} Q^{-1}\left(\frac{1-\beta q}{p}, \frac{t - t_0}{T_c} + Q\left(\frac{1-\beta q}{p}, -V(\boldsymbol{x}_0)^p\right)\right)\right]^{\frac{1}{p}}$. Thus, from Remark 4, the settling-time function for system (1) complies $T(\boldsymbol{x}_0) \leq T_c \left[1 - Q\left(\frac{1-\beta q}{p}, -V(\boldsymbol{x}_0)^p\right)\right] \leq T_c$, $\forall \boldsymbol{x}_0 \in \mathbb{R}^n$. Hence, the origin of system (1) is predefined-time-stable, with a predefined-time T_c. □

The following corollary exposes how Theorem III.1 generalizes the results proposed in [10], [11], [14].

Corollary III.2 (See [14])**.** *If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $V(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$, and
(ii) $V(\boldsymbol{x}) \geq 0$ and,
(ii) any solution $\boldsymbol{x}(t)$ of (1) satisfies

$$\dot{V}(\boldsymbol{x}) \leq -\frac{1}{p T_c} \exp\left(V(\boldsymbol{x})^p\right) V(\boldsymbol{x})^{1-p} \tag{4}$$

for $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and, constants $T_c := T_c(\boldsymbol{\rho}) > 0$ and $0 < p \leq 1$.

Then, the origin of system (1) is predefined-time-stable, with a predefined time T_c.

Proof. (See [14]) From the differential inequality (4), $V(\boldsymbol{x}(t))$ satisfies $V(\boldsymbol{x}(t)) \leq \left[\ln\left(\frac{1}{\frac{t-t_0}{T_c} + \exp(-V(\boldsymbol{x}_0)^p)}\right)\right]^{\frac{1}{p}}$. Thus, the settling-time function for the system (1) complies $T(\boldsymbol{x}_0) \leq T_c \left[1 - \exp(-V(\boldsymbol{x}_0)^p)\right] \leq T_c$, $\forall \boldsymbol{x}_0 \in \mathbb{R}^n$. Hence, the origin of system (1) is predefined-time-stable, with a predefined-time T_c.

Similarly, the proof of this corollary easily follows from Theorem III.1 with $\alpha = \beta = 1$ and $p = 1 - q$ and considering Remark 4. □

Remark 5. With a similar approach to the previous one, it follows the predefined-time characterization presented in [16].

Example 1. The system

$$\dot{x} = -\frac{\alpha^{\frac{\beta q-1}{p}}\Gamma\left(\frac{1-\beta q}{p}\right)}{mpT_c}\exp\left(\alpha\left|x\right|^{mp}\right)\left\lfloor x\right\rceil^{m(\beta q-1)+1} \tag{5}$$

where $x \in \mathbb{R}$, $\lfloor x\rceil = |x|\,\mathrm{sign}(x)$, and $m > 1$ is predefined-time stable, with a predefined-time T_c.

The result follows from considering the candidate Lyapunov function $V(x) = |x|^m$. Then $\dot{V}(x) = -\frac{\alpha^{\frac{\beta q-1}{p}}\Gamma\left(\frac{1-\beta q}{p}\right)}{pT_c}\exp\left(\alpha\left|x\right|^{mp}\right)|x|^{m\beta q} = -\frac{\alpha^{\frac{\beta q-1}{p}}\Gamma\left(\frac{1-\beta q}{p}\right)}{pT_c}\exp\left(\alpha V(x)^p\right)V(x)^{\beta q}$.

Remark 6. Consider the example $\dot{x} = -\exp(|x|)\lfloor x\rceil^{\frac{1}{2}}$ proposed in [20]. In this paper, it is shown that this system is fixed-time stable with a fixed time equal to $\sqrt{\pi}$. Firstly, note that the same results is obtained applying Theorem III.1 with $V(x) = |x|$, $\alpha = p = q = 1$, $\beta = 1/2$ which results in $T_c = \sqrt{\pi}$ since $\Gamma(1/2) = \sqrt{\pi}$. Secondly, the system in its current form does not provides a straightforward approach to select in advance the convergence time. Namely, how to change the time $\sqrt{\pi}$ to another value. For this reason, the given system can not be considered yet as predefined-time stable. However, this constraint is easily removed applying again Theorem III.1 with $V(x) = |x|$, $\alpha = p = q = 1$, $\beta = 1/2$ and $T_c > 0$. This procedure leads to the modified system $\dot{x} = -\frac{\sqrt{\pi}}{T_c}\exp(|x|)\lfloor x\rceil^{\frac{1}{2}}$, which is predefined-time stable, with a predefined time T_c. Finally, the modified system is a particular case of the system (5) presented in Example 1 with $\alpha = m = p = q = 1$, $\beta = 1/2$ and $T_c > 0$.

B. Predefined time stabilization of uncertain second-order systems

For this case, consider a system in the canonical form with a matched disturbance

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u + \Delta\end{aligned} \tag{6}$$

where $x_1, x_2, u, \Delta \in \mathbb{R}$ with $|\Delta| < \delta$.

The following theorem presents a controller which stabilizes system (6) in a predefined-time despite the disturbance term, that is $(x_1, x_2) = (x_1, \dot{x}_1) = (0, 0)$ for a given $T_c > 0$. Therefore, the proposed scheme is a second order sliding mode controller with predefined time convergence. The basis of the controller is the sliding variable proposed in [8]. The main advantage of this proposal is that despite the singularity presented in the dynamics of this variable, it is possible to design a nonsingular controller to stabilize it.

Theorem III.3. *Let the following control input for system* (6)*:*

$$\begin{aligned}u = &-\gamma_1^2\left(q_1 + p_1\left|x_1\right|^{p_1}\right)\left|x_1\right|^{q_1-1}\exp\left(\left|x_1\right|^{p_1}\right)\mathrm{sign}(\sigma)\\ &-\gamma_2\exp\left(\alpha_2\left|\sigma\right|^{p_2}\right)\lfloor\sigma\rceil^{\beta_2 q_2} - k\,\mathrm{sign}(\sigma)\end{aligned} \tag{7}$$

where $\sigma = x_2 + \left\lfloor \lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right\rceil^{\frac{1}{2}}$ *with the gains* $\gamma_1 = 2^{\frac{1-q_1/2}{p_1}}\Gamma\left(\frac{1-q_1/2}{p_1}\right)/p_1T_{c_1}$, $\gamma_2 = \alpha_2^{\frac{\beta_2 q_2-1}{p_2}}\Gamma\left(\frac{1-\beta_2 q_2}{p_2}\right)/p_2T_{c_2}$, *and* $k > \delta$ *depending on the parameters* $T_{c_1} > 0$, $p_1 > 0$, $1 \leq q_1 < 2$, $T_{c_2} > 0$, $\alpha_2 > 0$, $\beta_2 > 0$, $p_2 > 0$, $q_2 > 0$ *such that* $\beta_2 q_2 < 1$.

Therefore, the system (6) *closed by* (7) *is predefined-time stable, with a predefined time* $T_c = T_{c_1} + T_{c_2}$ *despite the disturbance term* Δ.

Proof. The dynamics of the variable σ is given by

$$\begin{aligned}
\dot{\sigma} &= u + \Delta + \\
&\frac{|x_2|(u+\Delta) + \gamma_1^2\left(q_1 + p_1|x_1|^{p_1}\right)|x_1|^{q_1-1}\exp\left(|x_1|^{p_1}\right)x_2}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}\\
&= -\gamma_1^2\left(q_1 + p_1|x_1|^{p_1}\right)|x_1|^{q_1-1}\exp\left(|x_1|^{p_1}\right)\mathrm{sign}(\sigma)\\
&- \gamma_2\exp\left(\alpha_2|\sigma|^{p_2}\right)\lfloor\sigma\rceil^{\beta_2 q_2} - k\,\mathrm{sign}(\sigma) + \Delta\\
&- \frac{|x_2|\,\gamma_2\exp\left(\alpha_2|\sigma|^{p_2}\right)\lfloor\sigma\rceil^{\beta_2 q_2}}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}\\
&- \frac{\gamma_1^2\left(q_1 + p_1|x_1|^{p_1}\right)|x_1|^{q_1-1}\exp\left(|x_1|^{p_1}\right)\left(|x_2|\,\mathrm{sign}(\sigma) - x_2\right)}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}\\
&- \frac{|x_2|\left(k\,\mathrm{sign}(\sigma) - \Delta\right)}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}
\end{aligned}$$

Thus, to analyze the stability of the variable σ, let the candidate Lyapunov function $V_2 = |\sigma|$. Then, for $\sigma \neq 0$

$$\begin{aligned}
\dot{V}_2 &= \dot{\sigma}\,\mathrm{sign}(\sigma)\\
&= -\gamma_1^2\left(q_1 + p_1|x_1|^{p_1}\right)|x_1|^{q_1-1}\exp\left(|x_1|^{p_1}\right)\\
&- \gamma_2\exp\left(\alpha_2|\sigma|^{p_2}\right)|\sigma|^{\beta_2 q_2} - k + \Delta\,\mathrm{sign}(\sigma)\\
&- \frac{|x_2|\,\gamma_2\exp\left(\alpha_2|\sigma|^{p_2}\right)|\sigma|^{\beta_2 q_2}}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}\\
&- \frac{\gamma_1^2\left(q_1 + p_1|x_1|^{p_1}\right)|x_1|^{q_1-1}\exp\left(|x_1|^{p_1}\right)\left(|x_2| - x_2\,\mathrm{sign}(\sigma)\right)}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}\\
&- \frac{|x_2|\left(k - \Delta\,\mathrm{sign}(\sigma)\right)}{\left|\lfloor x_2\rceil^2 + 2\gamma_1^2\exp\left(|x_1|^{p_1}\right)\lfloor x_1\rceil^{q_1}\right|^{\frac{1}{2}}}
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{V}_2 &\leq -\gamma_2\exp\left(\alpha_2|\sigma|^{p_2}\right)|\sigma|^{\beta_2 q_2} - k + \delta\\
&\leq -\frac{\alpha_2^{\frac{\beta_2 q_2-1}{p_2}}\Gamma\left(\frac{1-\beta_2 q_2}{p_2}\right)}{p_2T_{c_2}}\exp\left(\alpha_2 V_2^{p_2}\right)V_2^{\beta_2 q_2}
\end{aligned}$$

which implies the sliding mode $\sigma = 0$ for $t \geq T_{c_2}$.

In the sliding motion $\sigma = 0$ the system (6) reduces to

$$\dot{x}_1 = -\gamma_1\exp\left(\frac{1}{2}|x_1|^{p_1}\right)\lfloor x_1\rceil^{\frac{q_1}{2}}. \tag{8}$$

To analyze the stability of the sliding dynamics (8), let the Lyapunov function $V_1 = |x_1|$. Then, for $x_1 \neq 0$

$$\begin{aligned} \dot{V}_1 &= \operatorname{sign}(x_1)\dot{x}_1 \\ &= -\gamma_1 \exp\left(\frac{1}{2}|x_1|^{p_1}\right)|x_1|^{\frac{q_1}{2}} \\ &= -\frac{2^{\frac{1-q_1/2}{p_1}}\Gamma\left(\frac{1-q_1/2}{p_1}\right)}{p_1 T_{c_1}} \exp\left(\frac{1}{2}V_1^{p_1}\right)V_1^{\frac{q_1}{2}} \end{aligned} \tag{9}$$

which implies the sliding mode $x_1 = 0$, and as consequence of the variable σ, the sliding mode $x_2 = 0$ for $t \geq T_c = T_{c_1} + T_{c_2}$. □

Remark 7. Theorem III.1 is of paramount importance since it allows the construction of the second-order controller presented in Theorem III.3. For this case, the possibility to have the parameter q_1 larger than one avoids singularities in the control input (7) as x_1 reaches zero. In this form, the generalization allowed by Theorem III.1 extends the applicability of Corollary III.2, since the last does not provide a straightforward method to the predefined time stabilization of systems with order higher than one.

IV. Simulation Example

The simulations were programmed on Simulink in Matlab, based on the Euler integrator with 100KHz of sampling frequency. In order to show the reliability of the proposed controller, consider system (6) with $\Delta = 0$. The control parameters were set at $q_1 = 1.2$, $p_1 = 1$, $T_{c_1} = 1$, $q_2 = 0.5$, $p_2 = 1$, $T_{c_2} = 1$, $\alpha_2 = 1 \times 10^{-3}$ and $\beta_2 = 1$. From the simulation results of Figure 1, one can realize that the state converge to the origin before the predefined time $T_c = T_{c_1} + T_{c_2}$ for any initial condition.

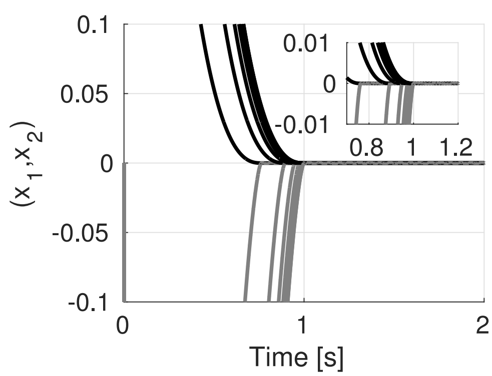

Fig. 1. Simulation results. $x_1(t)$ in black and $x_2(t)$ in gray, for different initial conditions.

V. Conclusion

This paper introduced a novel second order controller with predefined-time stability. First, a Lyapunov analysis that allows for the characterization and design of this controller was presented. Based on the theoretical basis provided by the proposed stability analysis, a second-order sliding mode controller was designed. The closed-loop system presents the practical advantage that the least upper bound for this settling time is known through an explicit and straightforward relationship with the controller gains.

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