An Equivalent Control Based Sliding Mode Observer Using High Order Uniform Robust Sliding Operators

Juan Diego Sánchez-Torres, Alexander G. Loukianov, Jaime A. Moreno and Sergey V. Drakunov

***Abstract*— In this paper a sliding-mode observer based on the equivalent control method for discontinuous functions for a class of non-linear systems is proposed. The observer structure and its existence conditions are presented. Besides, a class of high order sliding operators with the properties of uniform (w.r.t. initial conditions) finite time convergence and with reduction of chattering effect are exposed. The use of these operators in the observer design allows the calculation of the equivalent control and the observer convergence uniformly in finite time. A simulation example is presented to illustrate the proposed method.**

I. Introduction

Sliding mode approaches have been widely used for the problems of dynamic systems control and observation due to their characteristics of finite time convergence, robustness to uncertainties and insensitivity to external bounded disturbances [1], [2]. In observers based on sliding mode the sliding motion is obtained by means of a discontinuous term depending on the output error, into the controlling or observing system [3]. Additionally, by using the sign of the error to drive the sliding mode observer, the observer trajectories become insensitive to many forms of noise. Hence, some sliding mode observers have attractive properties similar to those of the Kalman filter (i.e. noise resilience) but with simpler implementation [4].

Several researchers have dealt with the issue of designing sliding-mode observers for different applications [5], including the classical problem of non-linear state estimation [6]. Note that the sliding modes techniques are based on the idea of the sliding manifold, that is an integral manifold with finite reaching time [7]. This manifold can be implemented by different methods including use of discontinuous function or continuous with discontinuous derivatives (so called higher order sliding modes). Let us note, that this issue of implementation, as demonstrated clearly by Utkin in [1] and earlier works is computational and depends on the system behavior in the boundary layer of the sliding manifold. Thus, the main difficulty and innovations in continuous-time sliding mode research is in the choice of the manifold rather than in the reaching phase that belongs more to numerical issue.

Indeed, once the sliding manifold $\sigma(x) = 0$ is chosen, the derivative(s) $\sigma^{(k)}$ of the function σ along the system trajectories can be expressed as function of control that has exactly same dimension as σ. Practically in all cases the sliding control and observers are implemented via digital computers, so, actually, discrete-time sliding mode is used, which is a version of a deadbeat control that makes σ to converge to zero in finite time. Let us note, that this algorithm can be dynamic, i.e. include past values of $\sigma(t_k)$ and in continuous-time will look as integrals of a function of σ.

In this work, our purpose is to discuss an observer design based over the equivalent control method for a class of observable SISO systems with some matching conditions on the input. This method has been treated previously in [3] and [8]. Subsequently, for the calculation of equivalent control, we propose the use of uniform exact high order sliding mode differentiators [9], [10]; and, the for observer convergence, the use of the Generalized Super-Twisting Algorithm [11], [12] is proposed. These high order sliding operators provides the observer with the properties of uniform (w.r.t. initial conditions) finite time convergence and with reduction of chattering effect.

In the following, in section II some mathematical preliminaries are given. In section III an observer design based over the equivalent control method for a class of observable SISO systems with some matching conditions on the input is proposed. An example of the proposed observer for synchronization of a chaotic system is presented in IV. Finally, in section V some conclusions are given.

II. Mathematical Preliminaries: Sliding Mode Observers Based on Equivalent Control Method

In this section, we present the mathematical concepts required to formulate an observer design based over the equivalent control method. Basic topics of sliding mode observers based on equivalent control method. These concepts are required for the observer design (and, by extension, to maintain its properties) to be proposed in section III.

Let us consider the following SISO system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output and f, g, h are sufficiently differentiable function

This work was supported by CONACyT under grant 129591.

Juan Diego Sánchez-Torres is with Automatic Control Laboratory, CINVESTAV-IPN Guadalajara, Av. del Bosque 1145 Col. El Bajío CP 45019, México dsanchez@gdl.cinvestav.mx

Alexander G. Loukianov is with CINVESTAV DEL I.P.N., Unidad Guadalajara; on sabbatical leave at Department of Electrical Engineering, DIELEC, Universidad de Guadalajara, Jalisco, Mexico, louk@gdl.cinvestav.mx

Jaime A. Moreno is with the Instituto de Ingeniería, Universidad Nacional Autónoma de México (UNAM), 04510 México D.F., México. JMorenoP@ii.unam.mx

Sergey V. Drakunov is with the Department of Physical Sciences, Embry-Riddle Aeronautical University, 600 S. Clyde Morris Blvd, Daytona Beach, FL 32114, USA drakunov@erau.edu

vectors.

For the system (1), let us define the vector of output derivatives, $H(x)$, as follows:

$$H(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_n(x) \end{pmatrix} = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}, \quad (2)$$

and the *Observability matrix,* $\mathcal{O}(x)$, as:

$$\mathcal{O}(x) = \frac{\partial H(x)}{\partial x} = \begin{pmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \end{pmatrix} \quad (3)$$

where $L_f^i h(x)$ represents the i-th Lie derivative of $h(x)$ in the f vector field direction. In addition, let us suppose the following condition:

Condition 1: The system (1) is globally observable, in the sense that the observability rank condition

$$\text{rank}(\mathcal{O}(x)) = n. \quad (4)$$

is fulfilled for all $x \in \mathbb{R}^n$.

Following the observer structure given in [3], for the case of systems without input, $g \equiv 0$, to estimate the state variables of the system (1) using the measurements y, an observer of the form

$$\dot{\hat{x}} = \mathcal{O}^{-1}(\hat{x}) M(\hat{x}) \mathcal{SL}\{V(t) - H(\hat{x})\} \quad (5)$$

is proposed, where $\mathcal{SL}\{\cdot\}$, is a causal operator ("Sliding Operator") such that $V(t) - H(\hat{x}) \to 0$ in finite time, i.e. for $t \geq t_1$, where $t_1 > 0$ is some time moment.

In (5), the operator $\mathcal{SL}\{\cdot\}$ have the form $\mathcal{SL}\{\xi\} = \text{col}(\mathcal{SL}\{\xi_1\}, \ldots, \mathcal{SL}\{\xi_n\})$, $M(\hat{x}) = \text{diag}(m_1(\hat{x}), \ldots, m_n(\hat{x}))$ is a $n \times n$ diagonal matrix with positive entries which are the gains of the observer and the vector $V(t)$ defined as

$$V(t) = \text{col}(v_1(t), \ldots, v_n(t)) \quad (6)$$

contains the system output and their first $n-1$ derivatives in the form $v_1(t) = y(t)$, $v_2(t) = \dot{y}(t)$, ... and $v_n(t) = y^{(n-1)}(t)$.

In the case of the sliding mode observer for the system with input, additional matching conditions are needed for the observation error to be independent of the input. So, addressing the relationship between system and inputs, the following condition is needed.

Condition 2: For any $x \in \mathbb{R}$, the vector

$$\frac{\partial H(x)}{\partial x} g(x)$$

does not depend on x, it means

$$\frac{\partial}{\partial x}\left[\frac{\partial H(x)}{\partial x} g(x)\right] = 0$$

for all $x \in \mathbb{R}$.

Under **Condition 2**, to estimate the state variables of the system (1) using the measurements y, an observer of the form

$$\dot{\hat{x}} = \mathcal{O}^{-1}(\hat{x}) M(\hat{x}) \mathcal{SL}\{V(t) - H(\hat{x})\} + g(\hat{x}) u \quad (7)$$

is used.

It is worth to note that the observer design consists of two problems: (i) the selection of the "Sliding Operator" $\mathcal{SL}\{\cdot\}$ such that $V(t) - H(\hat{x}) \to 0$ in finite time and (ii) the finite time calculation of the output derivatives vector $V(t)$. Drakunov [3] showed that using the structures (5) and (7), the recursive form

$$v_{i+i}(t) = \{\mathcal{SL}\{v_i(t) - h_i(\hat{x})\}\}_{eq} \quad (8)$$

with $i = 1, \ldots, n-1$, and with $\{\}_{eq}$ denoting an *equivalent value operator* of a discontinuous function in sliding mode [1] for the vector $V(t)$ calculation, a suitable choice of $M(\hat{x})$, $m_i(\hat{x})$ as an upper bound of $h_{i+1}(x)$ with $i = 1, \ldots, n-1$ and $m_n(\hat{x})$ of $L_f^n h(x)$, and the first order "Sliding Operator" $\mathcal{SL}$ defined by the $sign$ function, the observers in equations (5) and (7) converge in finite time, i.e. for $t \geq t_1$ the error $V(t) - H(\hat{x}) = 0$ and $v_{i+i}(t) = \{\mathcal{SL}\{v_i(t) - h_i(\hat{x})\}\}_{eq} = h_{i+1}(x)$ with $i = 1, \ldots, n-1$, and where $t_1 > 0$ is some time moment.

Let us note, that in [3] the $sign$ function simply was used as one of the ways enabling reachability of the desired sliding manifold $\sigma = V(t) - H(\hat{x}) = 0$, but the essence of the suggested observer idea was in the way the sliding manifold is designed. As was mentioned in the Introduction, the issue of eliminating chattering belongs to the boundary layer of the sliding manifold, i.e. how to implement "nonidealities" in the terminology of the fundamental work [1].

To eliminate chattering issues and, thus, obtain more accurate calculation of the equivalent control may be done using other means. In this paper we propose the use of high order uniform robust sliding operators with the aim to improve the calculation of the equivalent control and, in addition, to provide the observer with uniform convergence w.r.t. initial conditions [11], i.e. for $t \geq t_u$, where $t_u > 0$ is some time moment, $V(t) - H(\hat{x}) \to 0$ regardless of the initial conditions.

To ensure the uniform convergence of the observer w.r.t. initial conditions both, the "Sliding Operator" for the calculation of the output derivatives vector $V(t)$ and "Sliding Operator" which ensures $V(t) - H(\hat{x}) \to 0$ in finite time, must be uniform w.r.t. initial conditions. These two conditions are satisfied using the recently proposed "Exact and Uniformly Convergent Arbitrary Order Differentiator" [10] for the calculation of $V(t)$ and the "Generalized Super Twisting Algorithm" [11], [13] to ensure the convergence of $V(t) - H(\hat{x})$ to zero in finite time. The application of the two "Sliding Operators" is illustrated in the following section.

III. Sliding Mode Observer Using High Order Uniform Robust Sliding Operators

This section presents two "Sliding Operators": an uniformly convergent arbitrary order differentiator $\mathcal{SD}$ and the generalized super twisting algorithm, $\mathcal{GST}$. Besides, its applications in the calculation of equivalent control for improvement of sliding mode observer introduced in Section II in equations (5) and (7). Also, the stability analysis for the observer is presented.

A. Implementation of $\mathcal{SD}$ for $V(t)$ Calculation Via Uniformly Convergent Arbitrary-Order Differentiator

Real-time differentiation is a well-known problem, and between all the approaches that have been proposed to obtain time derivatives for a given signal, sliding mode based methods have demonstrated high accuracy and robustness for the calculation of higher order exact derivatives [9]. For this objective, an arbitrary-order differentiator based in a recursive scheme and which provides the best possible asymptotic accuracy in presence of input noises and discrete sampling is proposed in [14]. Based in this scheme, in [10] an uniformly convergent arbitrary order differentiator is introduced; we will use this differentiator as one of the possibilities for $\mathcal{SD}$ for the calculation of the vector $V(t)$.

Let the system output $y(t) \in \mathcal{C}^{\bar{n}}[0,\infty)$ be a function to be differentiated and let $n \leq \bar{n}$, then the n-th order differentiator is defined in two parts as follows:

Part 1: The observer must fulfills two proprieties, the first one is finite-time exactness, it means that error converges in finite-time despite the disturbance. This requirement is provided by the arbitrary-order differentiator [14]

$$\begin{aligned}
\dot{z}_0 &= \zeta_0, \\
\zeta_0 &= -\lambda_n L^{\frac{1}{n}} |z_0 - y|^{\frac{n-1}{n}} \operatorname{sign}(z_0 - y) + z_1 \\
\dot{z}_1 &= \zeta_1, \\
\zeta_1 &= -\lambda_{n-1} L^{\frac{1}{n-1}} |z_1 - \zeta_0|^{\frac{n-2}{n-1}} \operatorname{sign}(z_1 - \zeta_0) + z_2 \\
&\vdots \\
\dot{z}_{n-1} &= \zeta_{n-1}, \\
\zeta_{n-1} &= -\lambda_1 L^{\frac{1}{2}} |z_{n-1} - \zeta_{n-2}|^{\frac{1}{2}} \operatorname{sign}(z_{n-1} - \zeta_{n-2}) + z_n \\
\dot{z}_n &= -\lambda_0 L \operatorname{sign}(z_n - \zeta_{n-1})
\end{aligned} \tag{9}$$

where z_i is the estimation of the true signal $y^{(i)}(t)$. The differentiator provides finite time exact estimation under ideal condition when neither noise nor sampling are present. The parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ are suggested for the construction of differentiators up to the 5-th order. For the gain L case, the following condition is provided:

Condition 3: The $n+1$-th Lie derivative of $h(x)$ in the f vector field direction, $L_f^{n+1} h(x)$, must be bounded by the constant $L > 0$.

Part 2: The uniform convergence with respect to initial condition, i.e. the convergence time of the differentiator will be uniformly bounded by a constant which does not depends on the initial observation error. This requirement is provided by the practical uniform convergent differentiator [10]

$$\begin{aligned}
\dot{z}_0 &= -k_0 |z_0 - y|^{\frac{n+1+\alpha}{n+1}} \operatorname{sign}(z_0 - y) + z_1, \\
\dot{z}_1 &= -k_1 |z_0 - y|^{\frac{n+1+2\alpha}{n+1}} \operatorname{sign}(z_0 - y) + z_2, \\
&\vdots \\
\dot{z}_{n-1} &= -k_{n-1} |z_0 - y|^{\frac{n+1+n\alpha}{n+1}} \operatorname{sign}(z_0 - y) + z_n, \\
\dot{z}_n &= -k_n |z_0 - y|^{1+\alpha} \operatorname{sign}(z_0 - y)
\end{aligned} \tag{10}$$

where $k_0, \ldots, k_n$ are gains to be selected, with $\alpha > 0$ small and the gains selected such as the matrix

$$\mathbf{A} = \begin{bmatrix} -k_0 & 1 & 0 & \ldots & 0 \\ -k_1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_n & 0 & \ldots & 0 & 0 \end{bmatrix}$$

is Hurwitz, then error converges uniformly in the initial condition to a compact set $\mathcal{A}$. Finally, to obtain an uniform exact finite-time convergent differentiator, the finite time differentiator (9) can be combined with the practical uniform convergent differentiator (10). Due that both differentiators converge independently, this combination can be done in several ways [10]. For this case, the convergence time to $\mathcal{A}$ was calculated via simulation. Once a time (10) drives the system to $\mathcal{A}$, it is switched to (9).

See [14] and [10] for further details on the estimation of time of convergence, the error bounds for the signal $y(t)$ and their derivatives in presence of noise or discrete sampling and other properties and constraints of the differentiator.

B. The Super-Twisting algorithm based “Sliding Operator” $\mathcal{SL}$ Design

Another way to implement a $\mathcal{SL}$, and for this case, in order to enabling reachability of the desired sliding manifold $\sigma = 0$ and, in addition, ensure the uniform convergence of the observer w.r.t. initial conditions, is the use second order sliding mode algorithm for control and observation is the so-called “Generalized Super-Twisting Algorithm” $\mathcal{GST}$ [11], [13]

$$\varphi(\sigma) = \psi_1(\sigma) + \psi_2(\sigma), \tag{11}$$

where

$$\begin{aligned}
\psi_1(\sigma) &= -M_1 \phi_1(\sigma) \\
\dot{\psi}_2(\sigma) &= -M_2 \phi_2(\sigma)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\phi_1(\sigma) &= \mu_1 |\sigma|^{\frac{1}{2}} \operatorname{sign}(\sigma) + \mu_2 |\sigma|^{\frac{3}{2}} \operatorname{sign}(\sigma) \\
\phi_2(\sigma) &= \frac{\mu_1^2}{2} \operatorname{sign}(\sigma) + 2\mu_2^2 \sigma + \frac{3}{2}\mu_1\mu_2 |\sigma|^2 \operatorname{sign}(\sigma)
\end{aligned} \tag{13}$$

with $\mu_1 \geq 0$, $\mu_2 \geq 0$ scalars, $M_1 = \operatorname{diag}(m_{1,1}, \ldots, m_{1,n})$ and $M_2 = \operatorname{diag}(m_{2,1}, \ldots, m_{2,n})$ are two $n \times n$ diagonal matrices with positive entries which are the gains of the observer, the function $(\bullet)^{\frac{1}{2}}$ is extended to the form $\xi^{\frac{1}{2}} = \operatorname{col}\left(\xi_1^{\frac{1}{2}}, \ldots, \xi_n^{\frac{1}{2}}\right)$ for the expression $|\sigma|^{\frac{1}{2}}$, a similar definition for $(\bullet)^{\frac{3}{2}}$ and in the expressions of the form $|\sigma|^p \text{sign}(\sigma)$ the product is element to element.

Using the combination of the finite time differentiator (9) with the practical uniform convergent differentiator (10), the vector $V(t)$ results follows:

$$V(t) = \text{col}(y, z_1, \ldots, z_{n-1}) \tag{14}$$

instead of the recursive form (8) which generally use classic low-pass filters for the definition of the $\{\}_{eq}$ operator. Also, the vector $W(t)$ is defined as:

$$W(t) = \text{col}(z_1, \ldots, z_n) \tag{15}$$

Now, based on the equations (11)-(15), the following "Sliding Operator" is defined

$$\mathcal{SL}\{\sigma\} = W(t) + \psi_1(\sigma) + \psi_2(\sigma) \tag{16}$$

Remark 1: For the design of the "Sliding Operator" defined in (16), instead of the discontinuous terms (13) the following expressions could be used, regarding their respective limitations which in general are the lack of uniform convergence w.r.t. initial conditions and the chattering phenomena in (17).

- .The first order operator defined by the $sign$ function, as is shown [3]:

$$\begin{aligned} \phi_1(\sigma) =& \text{sign}(\sigma) \\ \phi_2(\sigma) =& 0 \end{aligned} \tag{17}$$

- The second order operator defined by the Classical Super-Twisting Algorithm [15]:

$$\begin{aligned} \phi_1(\sigma) =& |\sigma|^{\frac{1}{2}} \text{sign}(\sigma) \\ \phi_2(\sigma) =& \text{sign}(\sigma) \end{aligned} \tag{18}$$

- The second order operator defined by the Super-Twisting Algorithm with Linear Terms [16]:

$$\begin{aligned} \phi_1(\sigma) =& |\sigma|^{\frac{1}{2}} \text{sign}(\sigma) + M_1^{-1} M_3 \sigma - M_1^{-1} \psi_2(\sigma) \\ \phi_2(\sigma) =& \text{sign}(\sigma) + M_1^{-1} M_4 \sigma \end{aligned} \tag{19}$$

  with M_3 and M_4 defined in the same sense of M_1 and M_2.
- The second order operator defined by a first version of the Generalized Super-Twisting Algorithm [13]:

$$\begin{aligned} \phi_1(\sigma) =& \mu_1 |\sigma|^{\frac{1}{2}} \text{sign}(\sigma) + \mu_2 \sigma \\ \phi_2(\sigma) =& \frac{\mu_1^2}{2} \text{sign}(\sigma) + 2\mu_2^2 \sigma + \frac{3}{2} \mu_1 \mu_2 |\sigma|^{\frac{1}{2}} \text{sign}(\sigma) \end{aligned} \tag{20}$$

  with $\mu_1 \geq 0$, $\mu_2 \geq 0$ scalars.

C. Observer Design

For the system (1) the observer design procedure is defined in the following steps:

1) From the system equations (1), calculate the vector of output derivatives, $H(\hat{x})$.
2) Using the combination of the finite time differentiator (9) with the practical uniform convergent differentiator (10), calculate the vector $V(t)$ as in (14) and the vector $W(t)$ as in (15).
3) Based on the equations (11)-(16), define the "Sliding Operator"

$$\mathcal{SL}\{\sigma\} = W(t) + \psi_1(\sigma) + \psi_2(\sigma)$$

4) Finally, the resultant observer for the system (1) without input, $g \equiv 0$, is presented in equation (5).
   On the other hand, to estimate the state variables of the system (1) with the input fulfilling the **Condition 2** and using the measurements y, the proposed observer have the form provided by equation (7).

D. Observer Convergence

For the observer (5) and (7), under the diffeomorphism defined by the vector $H(x)$ and the observability matrix $\mathcal{O}$, the modified observation error, e, can be written in the transformed states $e = H(x) - H(\hat{x})$, in particular

$$\dot{e} = \dot{H}(x) - \dot{H}(\hat{x}) \tag{21}$$

leading to

$$\dot{e} = \begin{bmatrix} \dot{h}_1(x) \\ \dot{h}_2(x) \\ \vdots \\ \dot{h}_i(x) \\ \vdots \\ \dot{h}_{n-1}(x) \\ \dot{h}_n(x) \end{bmatrix} - [W(t) + \psi_1 + \psi_2] \tag{22}$$

that is

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_i \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} \dot{h}_1(x) - z_1 - \psi_{1,1} - \psi_{2,1} \\ \dot{h}_2(x) - z_2 - \psi_{1,2} - \psi_{2,2} \\ \vdots \\ \dot{h}_i(x) - z_i - \psi_{1,i} - \psi_{2,i} \\ \vdots \\ \dot{h}_{n-1}(x) - z_{n-1} - \psi_{1,n-1} - \psi_{2,n-1} \\ \dot{h}_n(x) - z_n - \psi_{1,n} - \psi_{2,n} \end{bmatrix} \tag{23}$$

where $\psi_{1,i} = m_{1,i}\phi_1(v_i(t) - h_i(\hat{x}))$ and $\dot{\psi}_{2,i} = m_{2,i}\phi_2(v_i(t) - h_i(\hat{x}))$ with $i = 1, \ldots, n$.

The convergence of $V(t)$ does not depend on the observer, but in **Condition 3**. Therefore, under the fulfillment of this condition, there is a time $t_d > 0$ (which not depends on initial error of the differentiator) such as if $t > t_d$, then $V(t) \equiv H(x)$ and $W(t) \equiv \dot{H}(x)$.

Then, for $t > t_d$, equation (23) becomes

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_i \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} -\psi_{1,1} - \psi_{2,1} \\ -\psi_{1,2} - \psi_{2,2} \\ \vdots \\ -\psi_{1,i} - \psi_{2,i} \\ \vdots \\ -\psi_{1,n-1} - \psi_{2,n-1} \\ -\psi_{1,n} - \psi_{2,n} \end{bmatrix} \tag{24}$$

where $\psi_{1,i} = m_{1,i}\phi_1(h_i(x) - h_i(\hat{x})) = m_{1,i}\phi_1(e_i)$ and $\dot{\psi}_{2,i} = m_{2,i}\phi_2(h_i(x) - h_i(\hat{x})) = m_{2,i}\phi_2(e_i)$ with $i = 1,\ldots,n$.

Let us note that the equation (24) is in the trivial form of the Super-Twisting. Therefore, with a suitable choice of the gain L and the matrices M_1, M_2, the system (24) is Globally Uniformly Finite Time Stable, i.e, the convergence of the observation error system (24) to zero is achieved in a finite time $t_{obs} > t_d$ which does not depends on the initial conditions and every selection of the matrices M_1, M_2 with positive elements on the diagonal.

Remark 2: A very similar convergence analysis can be done if for the design of the "Sliding Operator" defined in (16), instead of the discontinuous terms (13) the terms defined from (17) to (20) are used. Due to the lack of uniform convergence w.r.t. initial conditions of this operators only Globally Finite Time Stable can be demonstrated.

Remark 3: The use of a sliding mode differentiator provides two important properties:

1) For the calculation of the vector $V(t)$, in contrast to (8), the expressions does not depends on $H(\hat{x})$ yielding that the stability of the solutions for $V(t)$ does not depends on the observer.
2) The sliding mode differentiator allows the finite time exact calculation of the vector $W(t)$. When it is included in the "Sliding Operator" defined in (16) results on the independence of the matrix gains M_1, M_2 of $\hat{x}$ as is shown in equation (23), providing Global Stability to the observer.

Remark 4: Every bounded and observable SISO linear system fulfills from **Condition 1** to **Condition 3**. Giving a straight application to the proposed observer.

IV. Application Case: Rössler Chaotic System Synchronization

We highlight in this section the utility and the advantages of the observer design based over the equivalent control method of the previous recalls in the resolution of the observation problem. At first, we use a system an observer for synchronization of a chaotic system previously treated in the literature [17]. The problem of synchronization of chaotic systems can be seen as an observer design problem. The *Rössler system* [18] is a system conformed by three non-linear ordinary differential equations. For a given set of parameters, these differential equations define a continuous-time dynamical system that exhibits chaotic behavior. The set of equations (25) shows a state representation of the Rössler system. Let the forced Rössler system, defined as follows

$$\begin{aligned}\dot{x}_1 &= ax_1 + x_2\\ \dot{x}_2 &= -x_1 - x_3\\ \dot{x}_3 &= b + x_3(x_2 - c) + u\end{aligned} \tag{25}$$

Assuming the state $y = x_1$ as the output, the SISO system representation of the Rössler system in the way of (1) is

$$f(x) = \begin{pmatrix} ax_1 + x_2 \\ -x_1 - x_3 \\ b + x_3(x_2 - c)\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1\end{pmatrix} u \tag{26}$$

and

$$h(x) = x_1. \tag{27}$$

Noting that *Condition 2* is fulfilled and using the equations (28) and (29), we have designed and observer with the form shown in (5) and with the "Sliding Operator" defined by the Generalized Super-Twisting Algorithm (13) and (16), its state will be denoted by $\hat{x}_{GSTA}$. The values of the system constants are $a = 0.2$, $b = 0.2$ and $c = 5.7$. The parameter values for the differentiator are $L = 455$, $\lambda_1 = 1.1$, $\lambda_2 = 1.5$, and $\lambda_4 = 3$. For comparison purposes, another observer based on the "Sliding Operator" defined by the Classic Super-Twisting Algorithm (18) and (16) is designed, its state will be denoted by $\hat{x}_{STA}$.

For both observers, from (26) and (27) the Lie derivatives of the output are:

$$\begin{aligned} h &= x_1\\ L_f h &= ax_1 + x_2\\ L_f^2 h &= a(ax_1 + x_2) - x_1 - x_3,\end{aligned} \tag{28}$$

the corresponding observability matrix is

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 - 1 & a & -1\end{pmatrix} \tag{29}$$

and the gain matrices are $M_1 = \text{diag}(10\,5\,5)$ and $M_2 = \frac{5}{2}M_1$.

For simulation, $u = 5 + \sin(10t)$, the initial conditions for this system were $x(0) = [5\,5\,10]^T$ and $\hat{x}_{GSTA}(0) = \hat{x}_{STA}(0) = [0\,0\,0]^T$. In addition, for numerical integration Euler method with a step of 10×-3 was used. Figures 1, 2 and 3 show the simulation results for the system (26).

The state $\hat{x}_{GSTA_1}$ converges to x_1 in finite time of 0.6 seconds, until state $\hat{x}_{STA_1}$ converges to x_1 in finite time of 1.1 seconds.

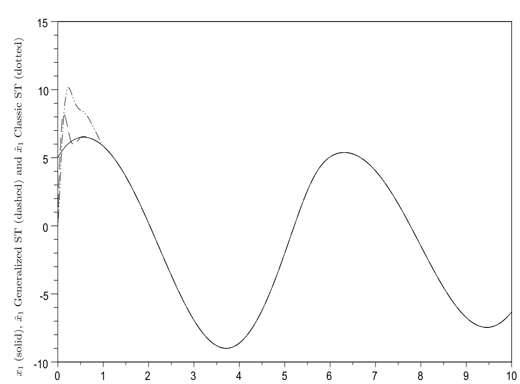

Fig. 1. Real x_1 (Solid), observed $\hat{x}_{GSTA_1}$ (Dashed) and observed $\hat{x}_{STA_1}$ (Dotted) states for system (25)

Then $\hat{x}_{GSTA_2}$ reaches to x_2 in finite time of about 1.6 seconds and $\hat{x}_{STA_2}$ converges to x_2 in finite time of 2.8 seconds. Note that $\hat{x}_{GSTA_2}$ and $\hat{x}_{STA_2}$ reach x_2 only after $\hat{x}_{GSTA_1}$ and $\hat{x}_{STA_1}$ converges to their respective states.

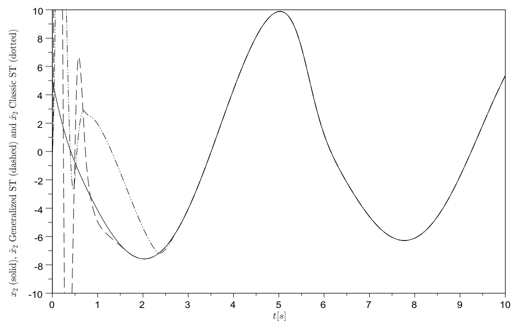

Fig. 2. Real x_2 (Solid), observed $\hat{x}_{GSTA_2}$ (Dashed) and observed $\hat{x}_{STA_2}$ (Dotted) states for system (25)

Finally, at a time of 1.7 seconds, $\hat{x}_{GSTA_3}$ converges to x_3 and $\hat{x}_{STA_3}$ converges to x_3 at a time of 3.1 seconds.

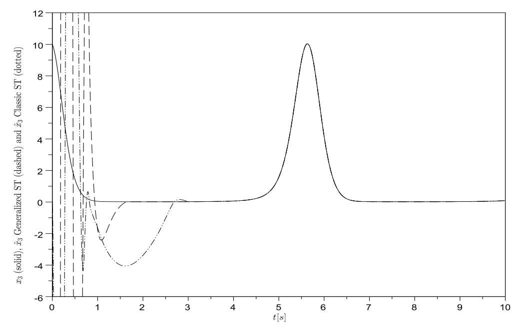

Fig. 3. Real x_3 (Solid), observed $\hat{x}_{GSTA_3}$ (Dashed) and observed $\hat{x}_{STA_3}$ (Dotted) states for system (25)

In the results shown in the figures 1, 2 and 3, both observers have exact finite time convergence.

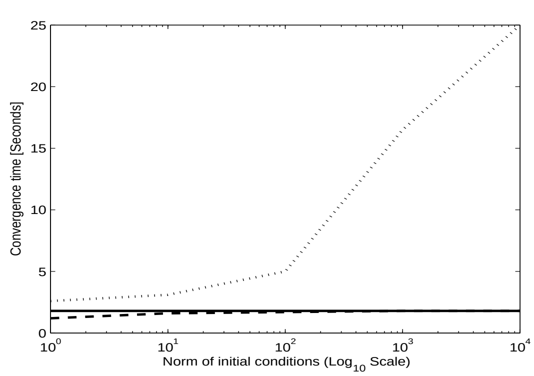

Fig. 4. Convergence time of both observers by growing initial condition norm

However, as shown in figure 4, the convergence time of the observer based on the Classic Super-Twisting Algorithm (dotted line) grows unboundedly with the norm of the initial condition, while the convergence time of the observer based on the Generalized Super-Twisting Algorithm (dashed line) is asymptotically bounded by a constant for growing norm of the initial condition (solid line). It could be shown that the terms defined from (17) to (20) have the similar unbounded behavior of Classic Super-Twisting Algorithm.

V. Conclusion

We show in this paper the design of a high order sliding mode observer for nonlinear systems using the equivalent control method. The generalized super-twisting algorithm and the uniform exact sliding mode differentiator were employed in order to ensure uniform finite time convergence of the observer and the reduction of chattering effect.

References

[1] V. Utkin, *Sliding Modes in Control and Optimization*. Springer Verlag, 1992.

[2] R. A. DeCarlo, S. Zak, and S. V. Drakunov, *The Control Handbook: a Volume in the Electrical Engineering Handbook Series, Chapter 50, Variable Structure, Sliding Mode Controller Design,*. CRC Press, Inc., 2011.

[3] S. V. Drakunov, “Sliding-mode observers based on equivalent control method,” in *Proc. 31st IEEE Conf. Decision and Control*, 1992, pp. 2368–2369.

[4] ——, “An adaptive quasioptimal filter with discontinuous parameter,” *Automation and Remote Control*, vol. 44, no. 9, pp. 1167–1175, 1983.

[5] C. M. Walcott, B.L. and S. Zak, “Comparative study of nonlinear state observation techniques,” *Int. J. Control*, vol. 45, pp. 2109–2132, 1987.

[6] J.-J. E. Slotine, J. K. Hedrick, and E. A. Misawa, “Nonlinear state estimation using sliding observers,” in *Proc. 25th IEEE Conf. Decision and Control*, vol. 25, 1986, pp. 332–339.

[7] S. V. Drakunov and V. Utkin, “Sliding mode control in dynamic systems,” *Int. J. Control*, vol. 55, pp. 1029–1037, 1992.

[8] S. V. Drakunov and M. Reyhanoglu, “Hierarchical sliding mode observers for distributed parameter systems,” *Journal of Vibration and Control*, vol. 36, no. 1, pp. 1–13, 2010.

[9] A. Levant, “Robust exact differentiation via sliding mode technique,” *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.

[10] M. T. Angulo, J. A. Moreno, and L. Fridman, “An exact and uniformly convergent arbitrary order differentiator,” in *Proc. 50th IEEE Conf. Decision and Control CDC 2011*, 2011.

[11] E. Cruz-Zavala, J. A. Moreno, and L. M. Fridman, “Uniform robust exact differentiator,” in *Proc. 49th IEEE Conf. Decision and Control (CDC)*, 2010, pp. 102–107.

[12] E. Cruz-Zavala, J. A. Moreno, and L. Fridman, “Uniform second-order sliding mode observer for mechanical systems,” in *Proc. 11th Int Variable Structure Systems (VSS) Workshop*, 2010, pp. 14–19.

[13] J. A. Moreno, “A linear framework for the robust stability analysis of a generalized super-twisting algorithm,” in *Proc. Computing Science and Automatic Control,CCE,2009 6th Int Electrical Engineering Conf*, 2009, pp. 1–6.

[14] A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *Int. J. of Control*, vol. 76, no. 9/10, pp. 924–941, 2003, special issue on Sliding-Mode Control.

[15] ——, “Sliding order and sliding accuracy in sliding mode control,” *International Journal of Control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[16] J. A. Moreno and M. Osorio, “A Lyapunov approach to second-order sliding mode controllers and observers,” in *Proc. 47th IEEE Conf. Decision and Control CDC 2008*, 2008, pp. 2856–2861.

[17] H. Nijmeijer and I. M. Y. Mareels, “An observer looks at synchronization,” *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 44, no. 10, pp. 882–890, 1997.

[18] O. E. Rössler, “An equation for continuous chaos,” *Physics Letters*, vol. 57A, no. 5, p. 397398, 1976.